



FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University

Teze disertace
k získání vědeckého titulu “doktor věd”
ve skupině věd fyzikálně-matematických

Arithmetic of small elements in totally real fields

Komise pro obhajoby doktorských disertací v oboru Matematické struktury

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Místo a datum: Praha, 17. listopadu 2024

I'm grateful to all my colleagues and co-authors for the fun and productive times we've had working together, in particular, to the co-authors of the articles contained in this thesis: Giacomo Cherubini, Nicolas Daans, Alessandro Fazzari, Andrew Granville, Tomáš Hejda, Siu Hang Man, Mentzelos Melistas, Piotr Miska, Magdaléna Tinková, Pavlo Yatsyna. I also thank Subham Roy and Robin Visser for their very helpful comments on a draft of the thesis, and to Jakub Krásenský, Mentzelos Melistas, Martina Vaváčková, Pavlo Yatsyna, and Mikuláš Zindulka for comments and help with my survey paper that forms a part of the thesis. I'm thankful to all the members of our research group UFOCLAN, that was built thanks to the very generous grants from Czech Science Foundation GAČR (21-00420M) and Charles University (PRIMUS/20/SCI/002).

The biggest thanks go to my family, to Leila, Jasmi, and Káťa.

Summary

Quadratic forms and number fields are one of the most classical and important areas of number theory, as well as of mathematics as a whole. Their study dates back to ancient Babylon, India, and Greece, followed by significant results by giants such as Fermat, Euler, and Gauss, all the way to the modern works of Fields medallists Bhargava (2014), Venkatesh (2018), and Viazovska (2022).

This thesis presents a number of new results in the area, all of which build on or are motivated by the role of small elements in the arithmetic of totally real extensions of the rationals. In particular, we focus on their applications to universal quadratic forms and class numbers.

A quadratic form over a totally real field K is universal if it is totally positive definite and represents all the totally positive algebraic integers. The best-known example is the sum of four squares $x^2 + y^2 + z^2 + w^2$ that represents all the positive rational integers, i.e., is universal over $K = \mathbb{Q}$. In this case, Bhargava–Hanke classified all the universal forms in the course of the proof of their 290-Theorem; we sketch the ideas behind the argument in Section I.1.

Then we turn our attention to the case of general totally real number fields K . After brief preliminaries (Section I.2), in Section I.3 we discuss why universal forms exist over every K and what is known about forms of small rank.

Section I.4 addresses the first non-trivial case of real quadratic fields $K = \mathbb{Q}(\sqrt{D})$. We explain the influence of certain small, indecomposable, elements of K on ranks of universal forms, and how these elements are determined by the continued fraction of $\sqrt{D}$. This leads to Theorem 7 of Kala–Yatsyna–Žmija that for almost all D , the minimal ranks of universal forms grow with D .

Section I.5 returns to the case of higher degree number fields and discusses methods for studying indecomposables and universal forms over them. Section I.6 describes the work of Kala–Tinková on the simplest cubic fields that builds on these tools together with the study

of small lattice points in the positive octant of the Minkowski space. The consideration of the simplest cubic fields is partly motivated by continued fraction families of real quadratic fields, discussed in Section I.7, that were the key for the construction of consecutive real quadratic fields with very large class numbers by Cherubini–Fazzari–Granville–Kala–Yatsyna.

Section I.8 focuses on another influential problem in the area, the lifting problem concerning the universality of quadratic forms whose coefficients are restricted to lie in $\mathbb{Q}$ or other fixed number field. In Section I.9, we further consider the existence of universal forms over fields K of infinite degree over $\mathbb{Q}$. In them, the behavior of small elements is captured by the Northcott property, which was related to universal forms by Daans–Kala–Man.

Finally, Section I.10 concludes the thesis by connecting the individual articles comprising its second part to the preceding discussion.

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I. Overview

This thesis concerns the arithmetic of totally real number fields. The common general theme of my research has been that algebraic integers that are *small* in a suitable sense have a surprisingly large influence on the behavior of important objects such as universal quadratic forms and class numbers – and can be used to control and estimate their properties.

In this Overview, we will explain and motivate the basic objects of study, with a focus towards outlining the main results and unifying themes in the ten published articles that form the second part of the thesis:

- [1] V. Kala and P. Yatsyna. Sums of squares in S -integers. *New York J. Math.*, 26:1145–1154, 2020.
- [2] V. Kala and P. Yatsyna. Lifting problem for universal quadratic forms. *Adv. Math.*, 377:107497, 2021.
- [3] T. Hejda and V. Kala. Ternary quadratic forms representing a given arithmetic progression. *J. Number Theory*, 234:140–152, 2022.
- [4] V. Kala and M. Tinková. Universal quadratic forms, small norms and traces in families of number fields. *Int. Math. Res. Not. IMRN*, 2023(9):7541–7577, 2023.
- [5] V. Kala. Number fields without universal quadratic forms of small rank exist in most degrees. *Math. Proc. Cambridge Philos. Soc.*, 174:225–231, 2023.
- [6] V. Kala and P. Yatsyna. On Kitaoka’s conjecture and lifting problem for universal quadratic forms. *Bull. Lond. Math. Soc.*, 55:854–864, 2023.
- [7] G. Cherubini, A. Fazzari, A. Granville, V. Kala, and P. Yatsyna. Consecutive real quadratic fields with large class numbers. *Int. Math. Res. Not. IMRN*, (14):12052–12063, 2023.

- [8] V. Kala and P. Miska. On continued fraction partial quotients of square roots of primes. *J. Number Theory*, 253:215–234, 2023.
- [9] V. Kala and M. Melistas. Universal quadratic forms and Dedekind zeta functions. *Int. J. Number Theory*, 20:1833–1847, 2024.
- [10] N. Daans, V. Kala, and S. H. Man. Universal quadratic forms and Northcott property of infinite number fields. *J. Lond. Math. Soc.*, 110:e70022, 2024.

A large part of this Overview is based on my survey paper [Kal23, Sections 2, 5–8]; smaller parts are taken from my other papers and grant proposals without mentioning this later in the text.

I.1 The 15- and 290-Theorems

A significant portion of my work focuses on *universal quadratic forms*. In the basic case of the rational integers $\mathbb{Z}$, these are positive definite quadratic forms that represent all positive integers. The motivation for their study comes from famous results such as Lagrange’s four square theorem ($x^2 + y^2 + z^2 + w^2$ is universal), classification works of Ramanujan, Dickson, and Willerding on quaternary universal forms, and Bhargava–Hanke 290-Theorem (if a quadratic form represents $1, 2, 3, \dots, 290$, then it is universal). Let us start here with a gentle introduction to the basic concepts over the ring of integers $\mathbb{Z}$ with a view towards the 290-Theorem, that should be hopefully understandable also by non-experts.

Recall that a *quadratic form of rank r* (or an *r -ary quadratic form*) over $\mathbb{Z}$ is a polynomial

$$Q(x_1, \dots, x_r) = \sum_{1 \leq i \leq j \leq r} a_{ij} x_i x_j, \quad a_{ij} \in \mathbb{Z}. \quad (1)$$

Typically we require the form to be *positive definite*, meaning that $Q(v) > 0$ for all $v \in \mathbb{Z}^r$, $v \neq 0$.

We attach the *Gram matrix* to Q , given by

$$M = M_Q = \begin{pmatrix} a_{11} & \frac{1}{2}a_{12} & \cdots & \frac{1}{2}a_{1r} \\ \frac{1}{2}a_{12} & a_{22} & \cdots & \frac{1}{2}a_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2}a_{1r} & \frac{1}{2}a_{2r} & \cdots & a_{rr} \end{pmatrix}. \quad (2)$$

Taking $v \in \mathbb{Z}^r$ to be a column vector $(x_1, \dots, x_r)^t$ we have

$$Q(v) = v^t M v.$$

The quadratic form Q is called *classical* if all the entries of M are integers, i.e., if a_{ij} are even for all $i \neq j$.

Each quadratic form has an associated *bilinear form* B defined by

$$Q(v + w) = Q(v) + Q(w) + 2B(v, w), \quad v, w \in \mathbb{Z}^r.$$

A positive definite form satisfies the Cauchy–Schwarz inequality: For all v and w ,

$$Q(v)Q(w) \geq B(v, w)^2.$$

In the 90's, Conway, Miller, Schneeberger, and Simons, and then Bhargava and Hanke [Bha99, BH11] came up with the following fascinating criteria for universality.

Theorem 1. *Let Q be a positive definite quadratic form over $\mathbb{Z}$. Then:*

- (a) *(15-Theorem, Conway–Schneeberger, $\sim$ 1995) If Q is classical and represents the integers*

$$1, 2, 3, 5, 6, 7, 10, 14, \text{ and } 15,$$

then it is universal.

- (b) *(290-Theorem, Bhargava–Hanke, $\sim$ 2005 [BH11]) If Q represents the integers*

$$1, 2, 3, 5, 6, 7, 10, 13, 14, 15, 17, 19, 21, 22, 23, 26, \\ 29, 30, 31, 34, 35, 37, 42, 58, 93, 110, 145, 203, \text{ and } 290,$$

then it is universal.

- (c) *Both of these lists of integers are minimal in the sense that for each integer n in the list, there exists a corresponding quadratic form that represents all of $\mathbb{Z}_{>0} \setminus \{n\}$, but does not represent n .*

While the 15-Theorem in part (a) is not too hard to prove, the 290-Theorem in part (b) is very challenging, not only because of the large amount of computations needed.

There have been a number of further exciting developments related to universal quadratic forms over $\mathbb{Z}$. For example, the conjectural 451-Theorem by Rouse [Rou14] says that *if a positive definite form represents the integers $1, 3, 5, \dots, 451$, then it represents all odd positive integers*. This result has been proved only under the assumption that each of the ternary forms $x^2 + 2y^2 + 5z^2 + xz$, $x^2 + 3y^2 + 6z^2 + xy + 2yz$, $x^2 + 3y^2 + 7z^2 + xy + xz$ represents all odd positive integers (that seems very hard to establish).

Escalations

We give a sketch of Bhargava’s proof of the 15-Theorem. The idea is to “build up” a universal quadratic form Q by gradually adding variables.

In order for Q to be universal, it must represent 1, and so it must contain x^2 . Slightly more precisely, a linear change of variables does not change universality and gives us x^2 (this will be made more precise soon, once we discuss quadratic lattices).

Now x^2 is clearly not universal as it does not represent 2, hence Q must contain $2y^2$ (again after a change of variables). We get the form $x^2 + 2axy + 2y^2$, where the coefficient of xy is $2a$ because we require the form to be classical, and so the corresponding Gram matrix is

$$\begin{pmatrix} 1 & a \\ a & 2 \end{pmatrix}.$$

What are the possible values for a ? By the Cauchy–Schwarz inequality $1 \cdot 2 \geq a^2$, which leaves the possibilities $a = 0, 1, -1$ with

the corresponding Gram matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}.$$

The quadratic forms $x^2 + 2xy + 2y^2$ and $x^2 - 2xy + 2y^2$ are equivalent by the change of variables $y \mapsto -y$ so we can forget about the third matrix. As for the second matrix, we can reduce the quadratic form by changing variables:

$$x^2 + 2xy + 2y^2 = (x + y)^2 + y^2 = X^2 + Y^2.$$

Note that, in terms of matrices, the Gram matrix of the resulting form is $C^t M C$ for an invertible matrix C . It can be obtained from M by successively applying the same row and column operations:

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We see that after two steps of escalations (i.e., of systematically expanding the form by introducing additional variables), we have two candidate forms $x^2 + 2y^2$ and $x^2 + y^2$. Since they do not represent 5, respectively 3, we pass to the matrices

$$\begin{pmatrix} 1 & 0 & b_1 \\ 0 & 2 & c_1 \\ b_1 & c_1 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 0 & b_2 \\ 0 & 1 & c_2 \\ b_2 & c_2 & 3 \end{pmatrix}.$$

We again determine all possible values for the coefficients and reduce the forms, which leads to the following possibilities (this can

be done as an exercise by the reader):

$$\begin{aligned} \begin{pmatrix} 1 & & \\ & 1 & \\ & & d_1 \end{pmatrix}, & \quad d_1 = 1, 2, 3 \\ \begin{pmatrix} 1 & & \\ & 2 & \\ & & d_2 \end{pmatrix}, & \quad d_2 = 2, 3, 4, 5 \\ \begin{pmatrix} 1 & & \\ & 2 & 1 \\ & 1 & d_3 \end{pmatrix}, & \quad d_3 = 4, 5. \end{aligned}$$

Continuing this process for rank 4, we get 207 forms, 201 of which are universal. This can be proved by local methods and genus theory (i.e., by a suitable use of the local-global principle). The remaining 6 forms represent all but one integers. After adding one more variable, we get 1630 universal forms of rank 5.

This procedure showed that if Q is universal, then it contains one of the rank 4 or 5 forms obtained above (here we are being quite imprecise as to what “contains” means; in the language of lattices introduced below it will mean the existence of a corresponding sublattice). These are all universal, and so the converse implication also holds, i.e., any quadratic form that contains one of these forms is universal.

But in the process of escalations, we only considered representations of small integers:

rank	
1	1
2	1, 2
3	1, 2, 3, 5
4 & 5	1, 2, 3, ..., 15

Thus if Q represents the integers 1, 2, 3, ..., 15, then it is universal, proving the 15-Theorem.

Proof of 290-Theorem

The proof of the 290-Theorem, although similar, is much more complicated. First, there are more cases to be considered (we have to continue the escalations up to rank 7, which leaves us with approximately 20 000 cases). Second, proving universality is sometimes very non-trivial and uses tools such as theta series (modular forms). For more information, see the original papers [Bha99, BH11] or the surveys [Hah08, Moo].

I.2 Background definitions

Let us now turn to the main topics of the thesis, for which we need to collect the following background information.

Let K be a totally real number field of degree d with d distinct real embeddings $\sigma_1 = \text{id}, \dots, \sigma_d : K \hookrightarrow \mathbb{R}$. The ring of algebraic integers of K is $\mathcal{O}_K$. The norm and trace of an element $\alpha \in K$ are then $N(\alpha) = \sigma_1(\alpha) \cdots \sigma_d(\alpha)$ and $\text{Tr}(\alpha) = \sigma_1(\alpha) + \cdots + \sigma_d(\alpha)$.

For two elements $\alpha, \beta \in K$ we define $\alpha \succ \beta$ if $\sigma_i(\alpha) > \sigma_i(\beta)$ for all i . We say that α is totally positive if $\alpha \succ 0$; the set of totally positive integers in K is denoted $\mathcal{O}_K^+$.

Finally, an important notion in our study of universal forms and lattices is indecomposability: A totally positive integer $\alpha \in \mathcal{O}_K^+$ is *indecomposable* if it cannot be decomposed as a sum $\alpha = \beta + \gamma$ of two totally positive integers $\beta, \gamma \in \mathcal{O}_K^+$. For example, totally positive units (as well as other elements of small norm) are always indecomposable.

In order to carry out arguments such as the one concerning the escalations more easily, it is convenient to work with quadratic lattices as a slight generalization of quadratic forms.

Let (L, Q) be a *quadratic $\mathcal{O}_K$ -lattice*, i.e., L is a finitely generated torsion-free $\mathcal{O}_K$ -module equipped with a quadratic map $Q : L \rightarrow \mathcal{O}_K$. Note that to a quadratic form q of rank r corresponds the *free* quadratic lattice $(\mathcal{O}_K^r, q)$. When q is a *diagonal* quadratic form $q(x_1, \dots, x_r) = \sum_{i=1}^r a_i x_i^2$, then we denote it (as well as the corresponding free lattice) by $\langle a_1, \dots, a_r \rangle$.

We will always assume that L is *totally positive definite* and *integral*

in the sense that $Q(v) \in \mathcal{O}_K^+$ for all $v \in L, v \neq 0$. Further, L is *classical* if all the values of the associated bilinear form lie in $\mathcal{O}_K$. L is *universal over K* if it represents over $\mathcal{O}_K$ all totally positive algebraic integers $\alpha \in \mathcal{O}_K^+$.

Given functions f, g whose values are positive real numbers, we write $f \ll g$ (or $g \gg f$) if there is a constant $c > 0$ such that $f(x) < cg(x)$ for all x in the domain of the functions. We write $f \ll_P g$ (or $g \gg_P f$) to stress that the constant c may depend on the parameter(s) P .

For any undefined notions, see [Kal23] or [O'M73].

I.3 Existence of universal forms

Let us now turn our attention more generally to the questions of existence of universal forms and of their properties, such as possible ranks. (Our discussion always applies to quadratic lattices, even when we talk about quadratic forms for simplicity.)

The most natural candidate for a universal form would be the sum of squares. Unfortunately, it is almost never universal, for Siegel [Sie45] showed that a sum of squares is universal over $\mathcal{O}_K$ only for

- $K = \mathbb{Q}$ (when 4 squares suffice) and
- $K = \mathbb{Q}(\sqrt{5})$ (when 3 squares suffice [Maa41]).

The proof considers representations of units and indecomposables.

One thus has to consider more general quadratic forms and aim at various classification results. This has been the most successful in the quadratic case.

Theorem 2 ([CKR96, Theorem 1.1]). *If $K = \mathbb{Q}(\sqrt{D})$ has a ternary classical universal form (and D is squarefree), then $D = 2, 3$, or 5 . In total, there are 11 such forms; examples in the three cases are*

- $x^2 + y^2 + (2 + \sqrt{2})z^2$ for $D = 2$,
- $x^2 + y^2 + (2 + \sqrt{3})z^2$ for $D = 3$,

- $x^2 + y^2 + \frac{5+\sqrt{5}}{2}z^2$ for $D = 5$.

The best available result in this direction is:

Theorem 3 ([KKP22, Theorem 3.2]). *If $K = \mathbb{Q}(\sqrt{D})$ has a universal lattice of rank ≤ 7 (and D is squarefree), then*

$$D < (5762838677310720000000005)^2.$$

This result builds on [Kim99]; in fact, Kim–Kim–Park [KKP22] give more precise results, also separately for classical lattices. The proof is based on considering the sublattice representing $1, 2, \dots, 290$ (it must have rank at least 4 when D is large thanks to the 290-Theorem), and the sublattice representing $\lceil 1 \cdot \sqrt{D} \rceil + 1 \cdot \sqrt{D}, \lceil 2 \cdot \sqrt{D} \rceil + 2 \cdot \sqrt{D}, \dots, \lceil 290 \cdot \sqrt{D} \rceil + 290 \cdot \sqrt{D}$ (that also must have rank ≥ 4).

Note that there is an 8-ary universal form over each $\mathbb{Q}(\sqrt{n^2 - 1})$ (when $n^2 - 1$ is squarefree) [Kim00] that can be explicitly constructed.

Such results on determining the small possible ranks of universal lattices are motivated by Kitaoka’s conjecture.

Conjecture 4 (Kitaoka). *There are only finitely many totally real number fields K having a ternary universal form.*

The conjecture still remains open. However, B. M. Kim and Kala–Yatsyna [6] proved at least a weak version of the conjecture saying that *when the degree d of K is fixed, then there are only finitely many such fields K .*

Some further interesting results are [ČLS⁺19, Deu08, Deu09, Lee08, KTZ20, Sas09].

Despite these results limiting the ranks of universal forms, one can observe that they exist over every totally real number field. There are at least two ways of proving this:

a) Proceed by a direct construction generalizing [Kim00] (see Corollary 9 below).

b) Hsia–Kitaoka–Kneser [HKK78, Theorem 3] showed a local-global principle for representations of elements with sufficiently large norm by Q , provided that the rank of Q is at least 5. So one can:

Find a form Q_0 that represents everything locally over all the finite places. For example, $Q_0 = \langle 1, 1, 1, \alpha \rangle$ where $\alpha \succ 0$ has additive valuation 1 at each dyadic place works, for already $\langle 1, 1, 1 \rangle$ is locally universal at all non-dyadic places [O’M73, 92:1b], and at the dyadic places, one can use Beli’s theorem [Bel22, Theorem 2.1]. Alternatively, one can use Riehm’s (much older) theorem [Rie64, Theorem 7.4] thanks to which it suffices to make sure that all classes mod 2 are represented – which is easily arranged by adding extra variables.

If necessary, add variables to Q_0 to obtain Q of rank ≥ 5 , for which one can use the asymptotic local-global principle [HKK78, Theorem 3]. Finally, add extra variables to cover the (finitely many) square classes of elements of small norms that are not represented by Q .

It is easy to see that there is never a universal form of rank $r = 1$ or 2 (for local reasons). Moreover, when the degree d of K is odd, it quickly follows from Hilbert’s reciprocity law that there is no ternary universal form [EK97, Lemma 3].

I.4 Lower bounds on ranks via continued fractions

Surprisingly, it turns out that universal lattices can require arbitrarily large ranks.

Theorem 5 ([BK15, Theorem 1], [Kal16, Theorem 1.1]). *For any positive integer r , there are infinitely many real quadratic fields $\mathbb{Q}(\sqrt{D})$ that do not have a universal lattice of rank $\leq r$.*

The broad idea behind the proof is the following. In a universal lattice (L, Q) , construct a sublattice that must have rank $\geq r$, for example by arranging it to contain pairwise orthogonal vectors v_i representing suitable elements β_i .

We need some notation to explain more details. Let us consider a real quadratic field $K = \mathbb{Q}(\sqrt{D})$ with squarefree $D > 1$. For simplicity, let us assume $D \equiv 2, 3 \pmod{4}$ so that $\mathcal{O}_K = \mathbb{Z}[\sqrt{D}]$ (but everything that we discuss here also generalizes to the case $D \equiv 1 \pmod{4}$).

There are two embeddings $K \hookrightarrow \mathbb{R}$, the identity and

$$\alpha = a + b\sqrt{D} \mapsto \alpha' = a - b\sqrt{D}.$$

Thus α is totally positive if and only if $a + b\sqrt{D} > 0$ and $a - b\sqrt{D} > 0$. The norm of α is $N(\alpha) = \alpha\alpha' = a^2 - b^2D$, and its trace is $\text{Tr}(\alpha) = \alpha + \alpha' = 2a$.

The fundamental unit of a real quadratic field is given in terms of the continued fraction of $\sqrt{D}$, which is periodic

$$\begin{aligned} \sqrt{D} &= [u_0, \overline{u_1, \dots, u_s}] \\ &= [u_0, u_1, \dots, u_s, u_1, \dots, u_s, u_1, \dots] = u_0 + \frac{1}{u_1 + \frac{1}{u_2 + \dots}}, \end{aligned}$$

and we know that $u_0 = \lfloor \sqrt{D} \rfloor$ and $u_s = 2\lfloor \sqrt{D} \rfloor$.

Let

$$\frac{p_i}{q_i} = [u_0, \dots, u_i]$$

be the *convergents* of the continued fraction. By an abuse of terminology, the quadratic integers $\alpha_i = p_i + q_i\sqrt{D}$ will also be called *convergents*. The element α_{s-1} is the fundamental unit. In other words, it generates the group of units, which can be described as

$$\mathcal{O}_K^\times = \left\{ \pm \alpha_{s-1}^k \mid k \in \mathbb{Z} \right\}.$$

Dress–Scharlau [DS82, Theorems 2 and 3] determined all the indecomposables in $\mathbb{Q}(\sqrt{D})$ in terms of the convergents as follows: *The indecomposables α are precisely the semiconvergents, i.e., elements of the form*

$$\alpha = \alpha_{i,t} = \alpha_i + t\alpha_{i+1}, \quad i \geq -1 \text{ odd}, \quad 0 \leq t < u_{i+2},$$

and their conjugates.

Note that $\alpha_{i,u_{i+2}} = \alpha_i + u_{i+2}\alpha_{i+1} = \alpha_{i+2}$, so indecomposables with a fixed i form an arithmetic progression going from α_i to α_{i+2} .

Now we can discuss a more precise result in the direction of Theorem 5 was obtained by Kala–Tinková [4], with inspiration by earlier results of Yatsyna [Yat19].

Theorem 6 ([4, Sections 7.1 and 7.3]). Let $\sqrt{D} = [u_0, \overline{u_1}, \dots, \overline{u_s}]$ and

$$U = \begin{cases} \max(u_1, u_3, \dots, u_{s-1}), & \text{if } s \text{ is even,} \\ \sqrt{D}, & \text{if } s \text{ is odd.} \end{cases}$$

Let Q be a universal quadratic form over $\mathbb{Q}(\sqrt{D})$ of rank r .

a) If Q is classical, then $r \geq U/2$.

b) In general, $r \geq \sqrt{U}/2$ (assuming $U \geq 240$).

To prove the theorem, we want to use minimal vectors in a quadratic lattice (L, Q) , i.e., nonzero vectors v such that $Q(v)$ is minimal (in some suitable sense). This approach works best over $\mathbb{Z}$, so we need to obtain a $\mathbb{Z}$ -lattice. In general, if $[K : \mathbb{Q}] = d$ and L is an $\mathcal{O}_K$ -lattice of rank r , then L can be naturally viewed as a $\mathbb{Z}$ -lattice of rank rd . Indeed,

$$L = \mathcal{O}_K v_1 + \dots + \mathcal{O}_K v_{r-1} + A v_r$$

for some fractional ideal A . Now $\mathcal{O}_K$ and A are isomorphic to $\mathbb{Z}^d$ as $\mathbb{Z}$ -modules and hence we can identify $L \simeq \mathbb{Z}^{dr}$ as a $\mathbb{Z}$ -module.

We will consider the quadratic form $\text{Tr}(\delta Q)$ for a suitable δ . We choose δ to satisfy that

- δ is a totally positive element (for then $\text{Tr}(\delta Q)$ is positive definite), and
- $\text{Tr}(\delta Q(v)) \in \mathbb{Z}$ for any $v \in L$.

This naturally leads us to looking at the *codifferent*

$$\mathcal{O}_K^\vee = \{\delta \in K \mid \forall \alpha \in \mathcal{O}_K : \text{Tr}(\delta \alpha) \in \mathbb{Z}\}.$$

We next make the following observation: Let $\alpha \in \mathcal{O}_K^+$. If there exists $\delta \in \mathcal{O}_K^{\vee,+}$ such that $\text{Tr}(\delta \alpha) = 1$, then α is indecomposable. For if $\alpha = \beta + \gamma$ for $\beta, \gamma \in \mathcal{O}_K^+$, then

$$1 = \text{Tr}(\delta \alpha) = \text{Tr}(\delta \beta) + \text{Tr}(\delta \gamma) \geq 2.$$

Now we have what we need to prove Theorem 6.

Sketch of proof of Theorem 6.

Step 1. Let $U = u_{i+2}$ for some odd i and consider the indecomposables $\alpha_{i,t}$, $0 \leq t < U$. We define $\delta = -\frac{1}{2\sqrt{D}}\alpha'_{i+1}$. It can be checked directly that $\delta \in \mathcal{O}_K^\vee$ and δ is totally positive. Next we compute the trace of

$$\delta\alpha_{i,t} = -\frac{1}{2\sqrt{D}}(p_{i+1} - q_{i+1}\sqrt{D}) \cdot (p_i + q_i\sqrt{D}) - \frac{t\sqrt{D}}{2D}\alpha'_{i+1}\alpha_{i+1}.$$

Since $\alpha'_{i+1}\alpha_{i+1} = N(\alpha_{i+1}) \in \mathbb{Z}$, we have

$$\mathrm{Tr}(\delta\alpha_{i,t}) = p_i q_{i+1} - p_{i+1} q_i = (-1)^{i+1} = 1.$$

Step 2. Take a quadratic $\mathcal{O}_K$ -lattice (L, Q) representing all the indecomposables $\alpha_{i,t}$, $0 \leq t < U$, so that $Q(v_t) = \alpha_{i,t}$ for some $v_t \in L$. Then $(\mathbb{Z}^{2r}, \mathrm{Tr}(\delta Q))$ is a $\mathbb{Z}$ -lattice containing $2U$ vectors of length 1, namely $\pm v_t$, as

$$\mathrm{Tr}(\delta Q(\pm v_t)) = \mathrm{Tr}(\delta\alpha_{i,t}) = 1.$$

Observe that if Q is classical, then $\mathrm{Tr}(\delta Q)$ is also classical. Therefore by repeatedly splitting off 1 (see beginning of Section I.5 below) we get

$$\mathbb{Z}^{2r} = \langle 1 \rangle \perp \langle 1 \rangle \perp \cdots \perp \langle 1 \rangle \perp L',$$

where $\langle 1 \rangle$ is repeated U times in the diagonal part. Thus $2r \geq U$.

Step 3. If Q is non-classical, we use known bound on the number of length-one vectors: There are $\leq \max(r^2, 240)$ of them in a non-classical $\mathbb{Z}$ -lattice of rank r . $\square$

Summary

Denote $m(K)$ the minimal rank of a universal $\mathcal{O}_K$ -lattice over K and $m_{\mathrm{class}}(K)$ the minimal rank of a classical universal $\mathcal{O}_K$ -lattice.

For $K = \mathbb{Q}(\sqrt{D})$ with $\sqrt{D} = [u_0, \overline{u_1}, \dots, \overline{u_s}]$, we have [BK18, 4]

$$\begin{aligned} \frac{1}{2} \max(u_i)^{1/2} &\leq m(K) \leq 8 \sum_{i=1}^s u_i \ll \sqrt{D}(\log D)^2 \\ \frac{1}{2} \max(u_i) &\leq m_{\mathrm{class}}(K) \leq 8 \sum_{i=1}^s u_i \ll \sqrt{D}(\log D)^2 \end{aligned}$$

If the fundamental unit is not totally positive (i.e., s odd), this is not too bad: the lower bound is $D^{1/4}$ and $D^{1/2}$ for $m(K)$ and $m_{\text{class}}(K)$, respectively. In the case when the fundamental unit is totally positive (i.e., s even), there are arbitrarily large differences between the lower and upper bounds, e.g., for $\sqrt{D} = [u_0, 1, 1, \dots, 1, 2u_0]$, we get $1/2 \leq m_{\text{class}}(K) \leq 4s$. Obtaining better lower bounds would require including all the indecomposables, not just $\alpha_{i,t}$ for a fixed i .

However, Kala–Yatsyna–Žmija recently expanded on these results by showing that

Theorem 7 ([KYZ23, Theorem 1.1]). *Let $\varepsilon > 0$. For almost all squarefree $D > 0$, we have that*

$$m_{\text{class}}(\mathbb{Q}(\sqrt{D})) \gg_{\varepsilon} D^{\frac{1}{12}-\varepsilon} \quad \text{and} \quad m(\mathbb{Q}(\sqrt{D})) \gg_{\varepsilon} D^{\frac{1}{24}-\varepsilon}.$$

By “almost all” we mean that the set of such D has (natural) density 1 among the set of all squarefree $D > 0$.

Finally, let us mention an open problem. Thanks to Chan–Oh [CO23], we know that there exist finite *criterion sets*, that give analogues of the 15- and 290-Theorems over any number field. However, the corresponding bounds are explicitly known only for classical forms over $\mathbb{Q}(\sqrt{5})$ [Lee08], and determining them more generally seems to be quite hard. In a very recent work, Kala–Krásenský–Romeo [KKR24] obtain several exciting results in this direction.

I.5 General results

Let us now turn to fields of higher degree, and to several possible approaches for studying universal forms over them. Throughout this section, K thus denotes a totally real number field of degree $d = [K : \mathbb{Q}]$.

Using units

Let (L, Q) be a classical universal quadratic lattice. Using Gram–Schmidt orthogonalization, it is easy to see that each unit splits off,

and in particular $L = \langle 1 \rangle \perp L'$ for some lattice $L' \subset L$. Now any square of a unit is represented by $\langle 1 \rangle$, so it need not be represented by L' . But if $\varepsilon \in \mathcal{O}_K^{\times,+}$ is a unit which is not a square, then L' must represent ε and hence $L = \langle 1, \varepsilon \rangle \perp L''$ for some lattice $L'' \subset L$. Continuing like this leads to the following observation: *The rank of a classical universal lattice is always greater than or equal to $\#\mathcal{O}_K^{\times,+}/\mathcal{O}_K^{\times 2}$.*

Since K is totally real, there are $d-1$ fundamental units $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{d-1}$, which implies $\mathcal{O}_K^{\times,+} \simeq \mathbb{Z}^{d-1}$. We can distinguish the two extreme cases:

- No fundamental unit is totally positive. Then each totally positive unit is a square, i.e., $\mathcal{O}_K^{\times,+} = \mathcal{O}_K^{\times 2}$.
- All fundamental units are totally positive. Then $\mathcal{O}_K^{\times 2} \simeq (2\mathbb{Z})^{d-1}$ and $\#\mathcal{O}_K^{\times,+}/\mathcal{O}_K^{\times 2} = 2^{d-1}$.

In the general situation when k fundamental units are totally positive, the rank of a classical universal lattice is $\geq 2^k$. If $k \geq 2$, this proves a special case of Kitaoka's conjecture, i.e., that K has no ternary classical form. Not much is thus missing to prove the full conjecture, it would suffice to show the existence of a few indecomposables!

Also in the non-classical case, one can obtain a similar result using our new criterion from Theorem 20.

As we already mentioned, recall that Hilbert's reciprocity law implies a theorem of Earnest–Khosravani [EK97]: If d is odd, then there is no ternary universal lattice (for local reasons).

Bounds on indecomposables

It is not hard to show a general upper bound on the norm of an indecomposable (although surprisingly, this bound was discovered only very recently, even though this question is, e.g., formulated as [Nar04, Problem 53]).

Theorem 8 ([6, Theorem 5]). *Each indecomposable has norm $\leq \text{disc}_{K/\mathbb{Q}}$. In fact, if $N(\alpha) > \text{disc}_{K/\mathbb{Q}}$, then $\alpha \succ \beta^2$ for some non-zero $\beta \in \mathcal{O}_K$.*

Proof. Let us sketch the proof. Let α be an element of norm $N(\alpha) > \text{disc}_{K/\mathbb{Q}}$, and let $\sigma_1(\alpha), \dots, \sigma_d(\alpha)$ be its conjugates. By Minkowski's theorem, for a sufficiently small $\varepsilon > 0$ the box

$$\left\{ x \in \mathbb{R}^d : |x_i| \leq \sqrt{\sigma_i(\alpha)} - \varepsilon, i = 1, \dots, d \right\}$$

in the Minkowski space contains a non-zero element $\beta \in \mathcal{O}_K$. Then $\alpha \succ \beta^2$, and α is therefore decomposable. $\square$

Before proceeding further, note that we have already seen [Sie45] that typically not all totally positive integers are sums of squares, but we can ask: What is the smallest integer P such that *if an element is the sum of squares, then it is the sum of at most P squares?* This integer P is called the *Pythagoras number* of the ring $\mathcal{O}_K$ and is known to be always finite, but can be arbitrarily large [Sch80b] (cf. also [Pol18]). However, there is an upper bound for Pythagoras numbers of orders in number fields that depends only on the degree of the number field [2, Corollary 3.3].

In the case of real quadratic number fields $K = \mathbb{Q}(\sqrt{D})$ the Pythagoras number is always ≤ 5 , and this bound is sharp [Pet73]. In fact, one can show that $P(\mathcal{O}_K) = 3$ for $D = 2, 3, 5$ [Coh60, Sch80a] and determine all D for which $P(\mathcal{O}_K) = 4$ (as in [CP62]). For some further recent results, see [1, Kra22, KRS22, Raš23, Tin23a].

Thanks to Theorem 8 above, if we have an element of large norm, we can successively subtract squares from it until we are left with something of norm $\leq \text{disc}_{K/\mathbb{Q}}$. If we then rewrite the sum of squares as the sum of P squares, we obtain the following result:

Corollary 9 ([6, Theorem 6]). *The quadratic form*

$$\sum \alpha x_\alpha^2 + y_1^2 + \dots + y_P^2,$$

where we sum over all square classes of elements $\alpha \in \mathcal{O}_K^+$ with norm $N\alpha \leq \text{disc}_{K/\mathbb{Q}}$, is universal and has rank $\ll \text{disc}_{K/\mathbb{Q}} \cdot (\log \text{disc}_{K/\mathbb{Q}})^{d-1}$.

Elements of trace 1

In Section I.3 we saw lower bounds for ranks of universal quadratic lattices in terms of elements of trace 1 in the codifferent. Exactly the same result holds in general.

Theorem 10 ([Yat19], [4, Section 7.1]). *Assume that there are $\beta_1, \dots, \beta_u \in \mathcal{O}_K^+$, $\delta \in \mathcal{O}_K^{\vee,+}$ such that $\text{Tr}(\beta_i \delta) = 1$ for all i . Then*

$$m(K) \geq \frac{u}{d}, \quad m_{\text{class}}(K) \geq \frac{\sqrt{u}}{d}.$$

How to find such elements? There is no general way (after all, there may be no totally positive elements in the codifferent that have trace 1), so one may have to rely on explicit constructions (such as in the proof of Theorem 6 or 12). However, let us also briefly discuss a method which uses the Dedekind zeta function.

The *Dedekind zeta function* is defined as

$$\zeta_K(s) = \sum_{A \leq \mathcal{O}_K} \frac{1}{(NA)^s}, \quad \Re s > 1,$$

where the sum runs over all integral ideals A , and NA is the ideal norm. The series converges absolutely for $\Re s > 1$ and ζ_K has a meromorphic continuation to the entire complex plane with a simple pole at $s = 1$. It satisfies a functional equation which relates $\zeta_K(s)$ to $\zeta_K(1 - s)$.

For us, the important fact is that Siegel related the value $\zeta_K(-1)$ to elements of small trace [Sie69], [Zag76, §1].

Theorem 11 (Siegel's formula for $\zeta_K(-1)$ and functional equation). *Assume that K is a totally real field of degree $d = 2, 3, 5, 7$. Then*

$$\sum_{\substack{\alpha \in \mathcal{O}_K^{\vee,+} \\ \text{Tr} \alpha = 1}} \sigma\left((\alpha)(\mathcal{O}_K^{\vee})^{-1}\right) = \frac{1}{b_d} \left| \text{disc}_{K/\mathbb{Q}} \right|^{3/2} \left(\frac{-1}{4\pi} \right)^d \zeta_K(2)$$

for a suitable $b_d \in \mathbb{Q}$ (e.g., $b_2 = \frac{1}{240}$, $b_3 = -\frac{1}{504}$, ...). Here

$$\sigma(B) = \sum_{A|B} N(A).$$

A similar formula holds in each degree d , but as the degree grows, it will involve elements of large traces (roughly, of traces up to $d/6$).

Siegel’s formula plays a key role in the proof of Theorem 16 concerning the lifting problem.

Large ranks

Let us also summarize here the known results on the existence of number fields with large minimal rank $m(K)$. For quadratic fields, we have already seen this in Section I.3, and in the cubic case, this is originally due to Yatsyna [Yat19, Theorem 5], and we will establish this in Section I.6 below.

A natural idea for extending these results to higher degrees is to start with a field K with large $m(K)$ and to consider overfields $L \supset K$. This was first carried out for multiquadratic fields [KS19], and then extended to all fields of degrees divisible by 2 and 3 [5].

Finally, Man [Man24] (and then Kala–Man [KM24]) significantly strengthened the results over multiquadratic fields and, in particular, established analogue of Theorem 7 on density 1 of real quadratic fields requiring large ranks of universal forms.

I.6 Families of cubic fields

Over a fixed field, one can compute everything explicitly, e.g., there are finitely many totally positive elements α with norm $N(\alpha) \leq \text{disc}_{K/\mathbb{Q}}$ (up to multiplication by units), and we can check which ones are indecomposable. We can also compute the codifferent and check which elements have trace 1.

For all fields of a given degree d , the problem is much harder. For real quadratic fields $\mathbb{Q}(\sqrt{D})$, one can use continued fractions to determine indecomposables and estimate ranks of universal forms. We might attempt to use generalized continued fractions [Ber71, Sch00] for fields of a higher degree – they are much worse behaved but there are some ongoing works in connection with the Jacobi–Perron algorithm [RSK24, KST23]. Geometric generalized continued fractions [Kar13],

such as Klein polyhedra and sails, are also promising, since there is a close connection to indecomposables, as very recently investigated by Kala–Man [KM24].

Rather than working with all fields, it is however typically easier to focus on a suitable family of fields that share some relevant properties (such as the structure of units and indecomposables).

The simplest cubic fields

We describe first the family of totally real cubic fields introduced by Shanks [Sha74].

Let $K = \mathbb{Q}(\rho)$ where ρ is a root of the polynomial

$$f(x) = x^3 - ax^2 - (a+3)x - 1, \quad a \geq -1.$$

If we order the three roots ρ, ρ', ρ'' as $\rho > \rho'' > \rho'$, then they are of approximate sizes $\rho \approx a+1$, $\rho'' \approx 0$ and $\rho' \approx -1$. It is a useful fact that all the roots are units, and are permuted under the mapping $\alpha \mapsto \frac{-1}{1+\alpha}$. We thus see that the other two conjugates ρ' and ρ'' also belong to K , K is the splitting field of f , and the Galois group $\text{Gal}(K/\mathbb{Q}) \simeq \mathbb{Z}/3$ is cyclic.

Another consequence is that K has units of all signatures. The discriminant of the polynomial f equals $\text{disc}_f = (a^2 + 3a + 9)^2$. If $a^2 + 3a + 9$ is squarefree (which happens for a positive density of a), then $\mathcal{O}_K = \mathbb{Z}[\rho]$. The units are small, hence the regulator is also small, and the class number formula implies that the class number is large, roughly $\approx a^2$ (up to a logarithmic factor).

When we search for indecomposables in a totally real number field K , it is natural to consider K in the Minkowski space by the mapping

$$\sigma : K \hookrightarrow \mathbb{R}^d, \alpha \mapsto (\sigma_1(\alpha), \sigma_2(\alpha), \dots, \sigma_d(\alpha)).$$

For example, consider the situation in a real quadratic field $K = \mathbb{Q}(\sqrt{D})$ with a fundamental totally positive unit ε . We can multiply every totally positive element by a suitable unit to move it into the cone $\mathbb{R}_{\geq 0} \cdot 1 + \mathbb{R}_{> 0} \cdot \varepsilon$ spanned by 1 and ε . If $\beta \succ 1$ or $\beta \succ \varepsilon$, then it is

not indecomposable, so we can further restrict our attention to the parallelogram

$$[0, 1) \cdot 1 + [0, 1) \cdot \varepsilon = \{t_1 \cdot 1 + t_2 \cdot \varepsilon \mid t_1, t_2 \in [0, 1)\}.$$

The situation in totally real cubic fields is similar. The totally positive units form a discrete set located on the hyperboloid $\{(x, y, z) \in \mathbb{R}^3 \mid xyz = 1\}$ in the Minkowski space. Up to multiplication by units, each element is contained in the polyhedral cone

$$C = \mathbb{R}_{\geq 0} \cdot 1 + \mathbb{R}_{\geq 0} \cdot \varepsilon_1 + \mathbb{R}_{\geq 0} \cdot \varepsilon_2 + \mathbb{R}_{\geq 0} \cdot \varepsilon_1 \varepsilon_2,$$

where ε_1 and ε_2 generate the totally positive unit group. This is essentially the content of Shintani's unit theorem [Neu99, Thm (9.3)]. The cone C is the union of two “triangular” cones spanned by $1, \varepsilon_1, \varepsilon_2$ and $\varepsilon_1, \varepsilon_2, \varepsilon_1 \varepsilon_2$, respectively. Again, we can restrict our search for indecomposables to the parallelepipeds $[0, 1) \cdot 1 + [0, 1) \cdot \varepsilon_1 + [0, 1) \cdot \varepsilon_2$ and $[0, 1) \cdot \varepsilon_1 + [0, 1) \cdot \varepsilon_2 + [0, 1) \cdot \varepsilon_1 \varepsilon_2$.

Note that the considerations above are a precursor to the more systematic study of sails [KM24].

In the simplest cubic fields, this approach (described in more detail in [4, Section 4]) is explicit enough that we can determine all indecomposables.

Theorem 12 ([4, Theorem 1.2]). *Let $K = \mathbb{Q}(\rho)$ be a simplest cubic field such that $\mathcal{O}_K = \mathbb{Z}[\rho]$. Up to multiplication by units, all indecomposables are*

- 1
- $1 + \rho + \rho^2$
- $-v - w\rho + (v+1)\rho^2$, $0 \leq v \leq a$, $v(a+2)+1 \leq w \leq (v+1)(a+1)$,
a triangle with $\frac{a^2+3a+2}{2}$ indecomposables.

For the indecomposable $1 + \rho + \rho^2$, we have

$$\min \left\{ \text{Tr}(\delta(1 + \rho + \rho^2)) \mid \delta \in \mathcal{O}_K^{\vee,+} \right\} = 2.$$

For the indecomposables $\alpha = -v - w\rho + (v + 1)\rho^2$ in the triangle,

$$\min \left\{ \text{Tr}(\delta\alpha) \mid \delta \in \mathcal{O}_K^{\vee,+} \right\} = 1.$$

Corollary 13 ([4, Theorem 1.1]). *Let $K = \mathbb{Q}(\rho)$ be a simplest cubic field such that $\mathcal{O}_K = \mathbb{Z}[\rho]$. Then*

- *there exists a diagonal universal form of rank $\sim 3a^2$,*
- *any classical universal lattice has rank $\geq \frac{a^2}{6}$,*
- *any universal lattice has rank $\geq \frac{a}{3\sqrt{2}}$.*

Gil Muñoz and Tinková [GT24] extended these results to also cover some non-monogenic simplest cubic fields.

Tinková [Tin23a, Tin23b] further obtained similar results in other families of cubic fields.

I.7 Continued fraction families of real quadratic fields

The aforementioned families of cubic fields are directly inspired by similar families of real quadratic fields, i.e., in degree two. The two most well-known examples are:

- Yokoi's family $\mathbb{Q}(\sqrt{m^2 + 4})$. If $m = 2n + 1$ is odd, then the relevant continued fraction is

$$\frac{1 + \sqrt{(2n + 1)^2 + 4}}{2} = [n + 1, \overline{2n + 1}].$$

- Chowla's family $\mathbb{Q}(\sqrt{4m^2 + 1})$.

It is also easy to see that $\sqrt{n^2 - 1} = [n - 1, \overline{1, 2(n - 1)}]$.

The idea is to generalize this by considering families $\mathbb{Q}(\sqrt{D})$ where

$$\sqrt{D} = [u_0, \overline{u_1, u_2, \dots, u_{s-1}, 2u_0}]$$

with s and $u_1, \dots, u_{s-1}$ fixed. A necessary condition is that the sequence $u_1, \dots, u_{s-1}$ must be symmetric, i.e., $u_i = u_{s-i}$. It turns out that this condition is almost sufficient for the existence of D .

Theorem 14 ([Fri88, Theorem]). *Let $u_1, \dots, u_{s-1}$ be symmetric, and define the numbers q_i via:*

$$q_{i+1} = u_{i+1}q_i + q_{i-1}, \quad q_{-1} = 0, q_0 = 1.$$

(This will be the sequence of denominators of the convergents $\frac{p_i}{q_i}$, and it does not depend on u_0 .)

There are infinitely many squarefree positive integers $D \equiv 2, 3 \pmod{4}$ such that $\sqrt{D} = [k, u_1, \dots, u_{s-1}, 2k]$ if and only if q_{s-2} or $\frac{q_{s-2}^2 - (-1)^s}{q_{s-1}}$ is even (otherwise, there is no such D , even when we drop the condition “squarefree”).

In such a case, all D and k are given by

$$D = D(t) = at^2 + bt + c, \quad k = k(t) = et + f, \quad t \geq 1$$

for fixed integers a, b, c, e, f that can be explicitly given in terms of u_i .

There is a similar characterization for $D \equiv 1 \pmod{4}$ and the continued fraction expansion of $\frac{1+\sqrt{D}}{2}$ [HK91].

These families have a number of advantageous properties:

- The fundamental unit ε depends linearly on t .
- The class number is large, essentially $t/\log t$ by the class number formula (see [7, DK18, DL18]).
- Indecomposables behave nicely (as in the simplest cubic fields).

Since the continued fraction controls ranks of universal lattices as we saw in Section I.4, we gain a lot of information about $m(K)$ for fields in a given family arising as in Theorem 14 and, in particular, we get results akin to Corollary 13.

I.8 Lifting problem for universal forms

When can a quadratic form with coefficients in $\mathbb{Z}$ be universal over a larger number field K ? The answer to this question, which we call the

lifting problem, seems to be “very rarely”, at least for number fields of small degrees.

For K different from $\mathbb{Q}$ and $\mathbb{Q}(\sqrt{5})$, the sum of squares is not universal by Siegel’s theorem. In particular, there is no universal diagonal $\mathbb{Z}$ -form (for each diagonal $\mathbb{Z}$ -form is represented by the sum of sufficiently many squares).

Kala–Yatsyna [2] extended this to general forms over real quadratic fields, as conjectured by Deutsch [Deu08].

Theorem 15 ([2, Theorem 1.1]). *If $K \neq \mathbb{Q}(\sqrt{5})$ is a real quadratic field, then there is no universal $\mathbb{Z}$ -form over K .*

For the proof, we again want to work with “*minimal vectors*” of the corresponding quadratic $\mathcal{O}$ -lattice (L, Q) , as in the proof of Theorem 6. For a $\mathbb{Z}$ -lattice, they are the vectors v such that $Q(v)$ is the smallest represented positive integer. This does not make a good sense over $\mathcal{O}_K$, so we take a suitable $\delta \in \mathcal{O}_K^\vee$ and consider $\text{Tr}_{K/\mathbb{Q}}(\delta Q(v))$ (which is a positive integer). We then decompose this “twisted trace form” as a tensor product and use properties of “lattices of E-type” to control its minimal vectors.

More generally, in [2], we further used Siegel’s formula for special values of the Dedekind zeta-function (Theorem 11) to show the following.

Theorem 16 ([2, Theorem 1.2]). *If K is a totally real number field of degree $d = 2, 3, 4, 5, 7$ which has*

- *principal codifferent ideal, and*
- *a universal quadratic form with coefficients in $\mathbb{Z}$,*

then $K = \mathbb{Q}(\sqrt{5})$ or $K = \mathbb{Q}(\zeta_7 + \zeta_7^{-1})$, where $\zeta_7 = e^{2\pi i/7}$. The form $x^2 + y^2 + z^2$ is universal over $\mathbb{Q}(\sqrt{5})$, and $x^2 + y^2 + z^2 + w^2 + xy + xz + xw$ is universal over $\mathbb{Q}(\zeta_7 + \zeta_7^{-1})$.

The assumption that the codifferent ideal is principal is unavoidable given the methods of [2]; it is satisfied for example when the rings of integers $\mathcal{O}_K$ has a power basis, or when the class group of K is $(\mathbb{Z}/2)^k$.

Theorem 16 already inspired a number of follow-up articles. Besides some preliminary works [GT24, 9] (the latter of which partly replaced the degree restriction by the assumption $d \leq 43$), D. Kim–S. H. Lee [KL24] significantly strengthened these results to cover all cubic and biquadratic fields. Notable are also the considerations of the indefinite case by He–Hu–Xu [HHX23] and Xu–Zhang [XZ22].

The current state of art is again due to me and Yatsyna [KY24, Theorem 1.2]: We showed that in degrees $d \leq 5$ there are no further fields having a universal form with $\mathbb{Z}$ -coefficients. The idea behind the proof is to move from the representability of elements over $\mathcal{O}_K$ to the representability of quadratic $\mathbb{Z}$ -forms q of rank $\leq d$. As $d \leq 5$ and $2q$ is classical, we can use a well-known theorem of Mordell and Ko [Ko37] that $2q$ is the sum of squares of linear forms. This in turn implies that all elements of $2\mathcal{O}_K^+$ are sums of squares, a condition that can be checked by large computations (involving ten million number fields) [KY24, Theorem 1.1].

Besides this, Kala–D. Kim–S. H. Lee recently showed the following general lifting result.

Theorem 17 ([KKL24, Theorem 1.1]). *Let F be a totally real number field and $d \in \mathbb{Z}^+$. There are at most finitely many totally real fields $K \supset F$ with $[K : F] = d$ such that there is an $\mathcal{O}_F$ -lattice L such that $L \otimes \mathcal{O}_K$ is universal.*

We further obtained the first explicit results concerning lifting from a base field $F \neq \mathbb{Q}$, see [KKL24, Theorems 1.2 and 1.3].

I.9 Infinite extensions and Northcott property

Recently we have been interested in studying universal quadratic forms and lattices over algebraic extensions $K \subset \mathbb{Q}$ of *infinite degree*. Again the natural case to consider is that K is *totally real* in the sense that all elements of K have only real conjugates.

Depending on the field K , it is then not clear whether universal lattices exist! When $K = \mathbb{Q}^{\text{tr}}$, the field of all totally real numbers,

then every element of $\mathcal{O}_K^+$ is a square, i.e., the unary form X^2 is universal. However, considering the maximal real multiquadratic field $\mathbb{Q}^{\text{tr},2} = \mathbb{Q}(\sqrt{n} \mid n \in \mathbb{Z}^+)$, Daans–Kala–Man [10] proved that it does not admit any universal lattice.

The key to the proof is our novel connection to the Northcott property: For an algebraic integer α , define its *house* as $|\overline{\alpha}| = \max_i(|\alpha_i|)$, where $\alpha_1 = \alpha, \dots, \alpha_n$ are all the conjugates of α . We then say that $\mathcal{O}_K$ has the *Northcott property* (with respect to the house) if, for every $r \in \mathbb{R}$, there exist only finitely many $\alpha \in \mathcal{O}_K$ with $|\overline{\alpha}| < r$. Already in 1962, J. Robinson [Rob62] showed that the ring of integers of $\mathbb{Q}^{\text{tr},2}$ has the Northcott property, and concluded that the first order theory of this ring is undecidable. The term Northcott property was coined in 2001 by Bombieri–Zannier [BZ01] who established that certain abelian extensions have the Northcott property. Since then, its study and applications have become a highly active research area (due to its connections with decidability and Diophantine geometry).

Theorem 18 ([10, Theorem 1.2]). *Let K be a totally real infinite extension of $\mathbb{Q}$. If $\mathcal{O}_K$ has the Northcott property, then there exists no universal quadratic lattice over K .*

This theorem immediately applies to $K = \mathbb{Q}^{\text{tr},2}$, as Bombieri–Zannier proved that it has the Northcott property (and likewise, e.g., for the compositum of all totally real Galois extensions of $\mathbb{Q}$ of degree exactly q , where $q = p$ or p^2 for a prime p).

Further, Daans–Kala–Man–Widmer–Yatsyna [DKM⁺24] investigated how often do infinite extensions have universal lattices or Northcott property. There are uncountably many totally real fields K , and so we have to proceed topologically: As each totally real field can be embedded into $\mathbb{Q}^{\text{tr}}$, we consider the set $\mathcal{X}$ of all subfields of $\mathbb{Q}^{\text{tr}}$. We endow $\mathcal{X}$ with the *constructible topology*, i.e., the coarsest topology for which the sets $\{K \in \mathcal{X} \mid a \in K\}$ are both open and closed, for all $a \in \mathbb{Q}^{\text{tr}}$. With respect to this topology, we obtain the following result.

Theorem 19 ([DKM⁺24, Theorems 1 and 3]). *The sets of totally real fields*

a) which admit a universal quadratic lattice, or

*b) for which the ring of integers has the Northcott property
are both meager subsets of $\mathcal{X}$.*

Recall that a subset of a topological space is called *meager* if it is a countable union of nowhere dense subsets and *comeager* if its complement is meager. In a Baire space, such as $\mathcal{X}$, non-empty open sets are non-meager and comeager sets are dense. Therefore, meagerness is a natural notion of “smallness” in a Baire space. Our approach is inspired by [DF21] and [EMSW23], who considered meagerness in a similar topological space to study non-definability of rings of integers in infinite extensions of $\mathbb{Q}$.

The proof of Theorem 19a) is based on a new connection between ranks of universal lattices and square classes of totally positive units that has not been known even over number fields.

Theorem 20 ([DKM⁺24, Theorem 2]). *Let $K \in \mathcal{X}$ and $n \geq 2$. A totally positive definite quadratic $\mathcal{O}_K$ -lattice of rank n represents at most $2n - 2$ classes of $\mathcal{O}_K^{\times,+}/\mathcal{O}_K^{\times 2}$.*

This theorem is a significant extension of the well-known “splitting off units” for classical lattices, thanks to which a classical lattice of rank n can represent at most n unit square classes.

I.10 The individual articles

Let us conclude the Overview by explicitly commenting on the articles comprising the second part of the thesis within the context outlined above.

Quadratic forms over $\mathbb{Z}$

While universal quadratic forms over $\mathbb{Z}$ are completely understood thanks to the 15- and 290-Theorems (see Section I.1), there are many open problems on integers represented (for example) by ternary forms. One of them was tackled by Hejda–Kala [3]: We investigated the diagonal ternary forms that represent all elements of a given

arithmetic progression $px + \ell$, especially in the case when p is a prime and $1 \leq \ell < p$. We provided significant computational evidence for the surprising conjecture that such forms exist only for the primes $p = 2, 3, 5, 7$, and 101 .

- [3] T. Hejda and V. Kala. Ternary quadratic forms representing a given arithmetic progression. *J. Number Theory*, 234:140–152, 2022.

Ranks of universal lattices

One could list more of the articles in this section, but [4] and [5] most directly deal with ranks of universal lattices.

- [4] V. Kala and M. Tinková. Universal quadratic forms, small norms and traces in families of number fields. *Int. Math. Res. Not. IMRN*, 2023(9):7541–7577, 2023.
- [5] V. Kala. Number fields without universal quadratic forms of small rank exist in most degrees. *Math. Proc. Cambridge Philos. Soc.*, 174:225–231, 2023.

Kala–Tinková [4] initiated the detailed study of the geometry of the totally positive octant that led to the results on indecomposables and universal lattices described in Section I.6. And in the paper [5], we established a procedure for extending results on large ranks of universal lattices from a subfield to a larger field, and used it to extend the known results (such as [BK15, Kal16, 4, Yat19]) in degrees 2 and 3 to fields in all degrees divisible by 2 or 3.

Continued fractions and class numbers

The papers [7] and [8] concern the structure of continued fractions of $\sqrt{D}$ and its relation to class numbers of real quadratic fields $\mathbb{Q}(\sqrt{D})$, as described in Section I.7.

- [7] G. Cherubini, A. Fazzari, A. Granville, V. Kala, and P. Yatsyna. Consecutive real quadratic fields with large class numbers. *Int. Math. Res. Not. IMRN*, (14):12052–12063, 2023.
- [8] V. Kala and P. Miska. On continued fraction partial quotients of square roots of primes. *J. Number Theory*, 253:215–234, 2023.

While it was known that there are $\gg \sqrt{X}$ squarefree values of $D \in [1, X]$ such that the fundamental unit ε_D of $\mathbb{Q}(\sqrt{D})$ satisfies $\log \varepsilon_D \ll \log D$ (and so the corresponding class number is large by class number formula; see, e.g., [Hoo84] for an overview of related results), it was expected that these units and class numbers for different values of D behave independently. Surprisingly, in Cherubini–Fazzari–Granville–Kala–Yatsyna [7] we observed that this is not necessarily the case! Using certain families coming from continued fractions, we showed that, for each positive integer k , there are many values of n such that the class numbers of $\mathbb{Q}(\sqrt{n+1}), \mathbb{Q}(\sqrt{n+2}), \dots, \mathbb{Q}(\sqrt{n+k})$ are all essentially as large as possible (and their units are all $\ll D$).

In [8], Kala–Miska consider continued fractions of $\sqrt{p}$ and $\sqrt{2p}$, where p is a prime. We obtained several results on the structure of their periods and partial quotients and, among others, answered several open questions of Miska–Ulas [MU22] on the appearance of 1 as a partial quotient.

Lifting problem

The articles [1, 2, 6, 9] deal primarily with the lifting problem discussed in Section I.8.

- [1] V. Kala and P. Yatsyna. Sums of squares in S -integers. *New York J. Math.*, 26:1145–1154, 2020.
- [2] V. Kala and P. Yatsyna. Lifting problem for universal quadratic forms. *Adv. Math.*, 377:107497, 2021.
- [6] V. Kala and P. Yatsyna. On Kitaoka’s conjecture and lifting problem for universal quadratic forms. *Bull. Lond. Math. Soc.*, 55:854–864, 2023.

- [9] V. Kala and M. Melistas. Universal quadratic forms and Dedekind zeta functions. *Int. J. Number Theory*, 20:1833–1847, 2024.

In [2], Kala–Yatsyna introduced the problem and proved several foundational results on it, including Theorems 15 and 16.

When studying universal lattices, it is often natural to consider not only lattices that represent all of $\mathcal{O}_K^+$, but those that represent all of $m\mathcal{O}_K^+$ for a fixed positive integer m . The advantages of doing this are twofold. First of all, many of the results about universality directly generalize also to this setting. Second, if (L, Q) is a universal lattice, then $(L, 2Q)$ is a *classical* lattice that represents $2\mathcal{O}_K^+$. Sometimes it can thus be better to work with $(L, 2Q)$, so that one can apply results that concern only classical lattices, such as the splitting off units discussed above, or the theorem of Mordell and Ko on the representability of classical $\mathbb{Z}$ -forms as the sum of squares of linear forms.

This is the setting of [1] where Kala–Yatsyna focus on the question: For which pairs (m, D) are all the elements of $m\mathcal{O}_{\mathbb{Q}(\sqrt{D})}^+$ represented by the sum of squares? They obtain several explicit positive and negative results that were further refined by Raška [Raš23].

Kala–Yatsyna [6] further proved a first version of the “weak lifting theorem”, Theorem 17 that was then used in [KKL24] in the proof of its version stated above. They also established Theorem 8 and Corollary 9, as well as a weak version of Kitaoka’s conjecture 4.

Finally, Kala–Melistas [9] expanded on the use of Dedekind zeta-function to strengthen some of the general estimates that appeared in [2].

Infinite extensions and Northcott property

Daans–Kala–Man [10] established Theorem 18 on the non-existence of universal lattices over infinite extensions that have the Northcott property, as explained in Section I.9.

- [10] N. Daans, V. Kala, and S. H. Man. Universal quadratic forms

and Northcott property of infinite number fields. *J. Lond. Math. Soc.*, 110:e70022, 2024.

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