Akademie věd České republiky

Teze disertace
k získání vědeckého titulu "doktor věd"
ve skupině věd fyzikálně-matematických

Selected Topics in Integrable Systems and Related Structures

Komise pro obhajoby doktorských disertací
v oboru Matematické struktury

Jméno uchazeče	**doc. RNDr. Artur Sergyeyev, Ph.D.**
Pracoviště uchazeče	Matematický ústav Slezská univerzita v Opavě
Místo a datum	Praha 2019

Contents

Summary

In the present dissertation we deal with two closely interwoven subjects, integrable systems, with integrability meant in the sense of soliton theory, and associated structures, in particular, symmetries, conservation laws, recursion operators and Hamiltonian structures, including some results on the said structures for both interable and nonintegrable systems.

Namely, we present a number of results on symmetries and conservation laws on evolution partial differential equations and systems in two independent variables. Namely, for the Bakirov system (14) and equation (12) we present complete descriptions of their generalized symmetries. We also give a number of results on conservation laws for linear evolution equations in two independent variables. For two more systems, (21) and the Sasa–Satsuma system (22) we give their bihamiltonian representations that lead to infinite hierarchies of commuting symmetries and integrals of motion. We also give a new compact description for Hamiltonian structures compatible with a given nondegenerate finite-dimensional Hamiltonian structure, and present a version of this result applicable to a broad class of Hamiltonian structures associated with evolution systems in two independent variables.

As far as integrable systems per se are concerned, in this dissertation we address in a positive fashion a longstanding problem of search for integrable partial differential systems in the case of four independent variables (4D), perhaps most relevant for possible applications, especially in physics, given that according to general relativity our spacetime has dimension four. Namely, we present an entirely new broad class of 4D integrable systems with Lax pairs involving contact vector fields. In particular, we show that this class contains two infinite families of such systems, thus establishing that there is significantly more integrable 4D systems than it was long believed. We also present what is, to the best of our knowledge, a first example of a 4D integrable system with an algebraic (rather than rational) nonisospectral Lax pair.

1 Preface

The theory of integrable systems has its roots in trying to answer a simple and natural question: when an ordinary differential equation, or a system of such equations, can be integrated by quadratures? An important milestone here is the Liouville theorem [31] in classical mechanics giving a sufficient condition for this to occur for an important class of Hamiltonian systems that are, inter alia, of considerable significance for applications, see e.g. [3, 9, 39].

Another great breakthrough in the field of integrability has occurred about half a century ago with the discovery of the so-called inverse scattering transform (IST) in the seminal work [20]. In this paper it was shown that solving the Cauchy problem for the *nonlinear* Korteweg–de Vries (KdV) equation, a remarkable evolutionary partial differential equation in one dependent and two independent variables satisfied, inter alia, by a certain generating function for intersection numbers of complex curves arising in the Witten conjecture and its proof by Kontsevich, see e.g. [44], can, under certain conditions, be reduced to a sequence of *linear* problems, and the procedure in question became known as IST. This is possible because the KdV equation can be written as a compatibility condition for an overdetermined system of linear equations (such overdetermined systems in the context of integrable systems are called the Lax pairs or the Lax/Lax-type representations in honour of P.D. Lax who discovered the Lax pair for the KdV, see e.g. [1, 16] for details). An important consequence of the above is the construction of infinitely many explicit exact solutions of the KdV equation – the multisoliton solutions.

Moreover, existence of the Lax pair for the KdV paved the way to construction by S.P. Novikov et al. [14] of other important classes of explicit exact solutions for this equation, namely, the quasiperiodic finite-gap solutions that are inextricably related to algebraic geometry. Note that making use of the counterparts of these solutions for the KP equation, a natural integrable generalization of the KdV equation to the case of three independent variables, enabled Shiota [42] to prove the longstanding Schottky conjecture in algebraic geometry.

Yet another notable consequence of the presence of the Lax pair is existence of infinitely many nontrivial independent local conservation laws for the KdV equation, cf. e.g. [1] and references therein, which shows, inter alia, that the associated dynamics is highly regular rather than chaotic.

It was quickly realized that the KdV equation is by no means an isolated example – there is plenty of partial differential systems that admit ‘good’ Lax pairs from which infinite hierarchies of conservation laws can be extracted and that are, at least in principle, amenable to the IST. In what follows we shall refer to the partial differential systems with ‘good’ Lax pairs in the above sense as to the *integrable* ones. While that some authors use the term integrable (or C-integrable) also for the systems that can be linearized by an appropriate change of variables but in the present dissertation we shall not do that.

Note that soliton and multisoliton solutions for KdV and many other integrable systems, as well as other types of exact solutions constructed using the Lax pairs, like, e.g., the multi-instanton solutions for the (anti)self-dual Yang–Mills equations obtained using the Atiyah–Drinfeld–Hitchin–Manin construction [5], have found significant applications both in physics and in pure mathematics, see for example Donaldson’s revolutionary works on geometry of four-dimensional manifolds using instantons, cf. e.g. [4].

Integrable systems are well known to have a number of remarkable structures attached to them. These include Lax pairs, symmetries, conservation laws, Poisson structures and more, see e.g. [9, 12, 28, 39]. Exploring these structures and their properties can provide one with quite a bit of insight into the behavior of the systems under study and their properties, be these systems integrable or not. For one, the presence of large number of symmetries and/or conservation laws indicates that the system under study has a highly constrained, and hence likely quite regular, dynamics, cf. e.g. the discussion in [3, 9, 39].

For example, for certain classes of partial differential systems the presence of sufficiently many symmetries of certain kind can be employed for search and classification of integrable cases, cf. e.g. the survey [35] and references therein, as well as for establishing nonintegrability.

The rest of the dissertation is organized as follows. Section 2 provides a brief review of geometric approach to the study of ordinary and partial differential systems employing the jet bundle language. Section 3 presents a number of results on generalized symmetries, conservation laws and Hamiltonian structures for evolution systems. Section 4 presents two constructions for important objects associated to integrable systems, namely, hierarchies of commuting nonlocal symmetries and recursion operators. In Section 5 we present a construction of a novel broad class of integrable systems in four independent variables using a new kind of Lax pairs related to contact geometry. After that comes the list of papers constituting the core of the dissertation, the reference list, and the reprints of the papers constituting the said core.

2 Preliminaries

Following mostly [26, 27, 28] (cf. also e.g. [25] and references therein) we briefly recall here the basics of the geometric approach to partial differential systems.

For a smooth manifold M of dimension n and a vector bundle $\pi\colon E \to M$ of rank N consider the bundles of k-jets $\pi_k\colon J^k(\pi) \to M$, $k \geq 0$ with the natural projections $\pi_{k+1,k}\colon J^{k+1}(\pi) \to J^k(\pi)$.

The manifold of infinite jets $J^\infty(\pi)$ is then defined as the inverse limit with respect to the above projections, and we also can define the bundles $\pi_\infty\colon J^\infty(\pi) \to M$ and $\pi_{\infty,k}\colon J^\infty(\pi) \to J^k(\pi)$. For any section $s\colon M \to E$ of π its infinite jet $j_\infty(s)\colon M \to J^\infty(\pi)$ is a section of π_∞. We have the embeddings $\pi^*_{k+1,k}\colon C^\infty(J^k(\pi)) \to C^\infty(J^{k+1}(\pi))$, and define the algebra of smooth functions on $J^\infty(\pi)$ as $\mathcal{A}(\pi) = \cup_{k\geq 0} C^\infty(J^k(\pi))$.

One important geometric structure on $J^\infty(\pi)$ that we will need is the Cartan distribution $\mathcal{C}$: for any point $\theta \in J^\infty(\pi)$ we define the Cartan plane $\mathcal{C}_\theta$ as the tangent plane to the graph of an infinite jet passing through this point. The said distribution is formally integrable, that is, if X and Y are vector fields in $\mathcal{C}$ then the commutator $[X, Y]$ lies there as well. Every Cartan plane $\mathcal{C}_\theta$ is n-dimensional and projects isomorphically to $T_{\pi_\infty(\theta)}M$ by the differential of π_∞. For this reason, any vector field Z on M can be uniquely lifted to a vector field $\mathcal{C}_Z$ on $J^\infty(\pi)$. The correspondence $Z \mapsto \mathcal{C}_Z$ is $C^\infty(M)$-linear and preserves the commutator. Moreover, $\pi_{\infty,*}(\mathcal{C}_Z) = Z$.

This gets us a connection known as the Cartan connection. In the standard local coordinates $x^1, \ldots, x^n$, $u^1_\alpha, \ldots, u^N_\alpha$ in $J^\infty(\pi)$, α being symmetric multi-index consisting of the integers $1, \ldots, n$, the Cartan connection is determined by the correspondence

$$\mathcal{C}\colon \frac{\partial}{\partial x^i} \mapsto D_{x^i} = \frac{\partial}{\partial x^i} + \sum_{\alpha,A} u^A_{\alpha i} \frac{\partial}{\partial u^A_\alpha}, \tag{1}$$

where the fields D_{x^i} are called the total derivatives. Differential operators in total derivatives are called $\mathcal{C}$-differential operators.

To make contact with the standard setup for the study of partial differential systems, recall that x^i represent independent and u^A dependent variables. We shall occasionally use the notation $\boldsymbol{u} = (u^1, \ldots, u^N)^T$ and $\vec{x} = (x^1, \ldots, x^n)^T$, where the superscript T indicates matrix transposition. A partial differential system in this setting is a submanifold in $J^k(\pi)$. Locally, such a system can be given by the conditions $F_1 = \cdots = F_m = 0$, where F_I are smooth functions on $J^k(\pi)$.

Such smooth functions will hereinafter often be referred to as *local* functions. We shall sometimes use the notation $f(\vec{x}, [\boldsymbol{u}])$ to indicate that f is a local function.

The infinite prolongation of the system in question is the submanifold (in general, with singularities) $\mathcal{E}$ in $J^\infty(\pi)$ satisfying the conditions $(D_{x^{i_1}} \circ \cdots \circ D_{x^{i_s}})(F_I) = 0$ for all $I = 1, \ldots, m$, $s \geq 0$, and $1 \leq i_1, \ldots, i_s \leq n$. Such a prolongation is known as a *diffiety*.

Notation: nD indicates n independent variables a.k.a. n dimensions, e.g. 3D for $n = 3$ and 4D for $n = 4$. Note that in the literature, including the papers constituting the coire of the present dissertation, one also uses the terms $(n+1)$-dimensional or $n+1$ dimensions to indicate $n+1$ independent variables; while some authors use this notation to indicate that one of the independent variables is distinguished in some way but we do not really adhere to this convention here.

The Cartan connection can be restricted from the bundle π_∞ to the subbundle $\pi_\infty|_{\mathcal{E}} \colon \mathcal{E} \to M$ and so any $\mathcal{C}$-differential operator restricts from $J^\infty(\pi)$ to M. On the other hand, using (1) we can lift any linear differential operator on M to a $\mathcal{C}$-differential operator on $\mathcal{E}$. We always assume below that $\mathcal{E}$ is differentially connected which means that the only solutions of the system $D_{x^i}(f) = 0$, $i = 1, \ldots, n$, on $\mathcal{E}$ are constants.

In particular, let $\ell_{\mathcal{E}}$ denote the restriction to $\mathcal{E}$ of the linearization operator with the entries

$$\left(\sum_\alpha \frac{\partial F_I}{\partial u^A_\alpha} D_\alpha \right), \qquad I = 1, \ldots, m, \quad A = 1, \ldots, N.$$

Then the solutions of the equation

$$\ell_{\mathcal{E}}(\phi) = 0 \quad \text{on} \quad \mathcal{E} \tag{2}$$

are identified with *generalized*, or *(infinitesimal) higher*, *symmetries* of $\mathcal{E}$, i.e., with the vertical vector fields $\sum_A \phi^A \partial/\partial u^A$ whose infinite prolongations

$$\sum_{A,\alpha} D_\alpha(\phi^A)\partial/\partial u^A_\alpha$$

are tangential to $\mathcal{E}$.

The solutions of equation formally adjoint to (2), that is,

$$\ell^*_{\mathcal{E}}(\psi) = 0 \quad \text{on} \quad \mathcal{E}$$

are called *cosymmetries*; here $\ell^*_{\mathcal{E}}$ is the formal adjoint of $\ell_{\mathcal{E}}$.

The lift d_h of the de Rham differential gives rise to the horizontal de Rham complex on $\mathcal{E}$; d_h-closed $(n-1)$-forms are conservation laws of $\mathcal{E}$ and d_h-exact forms are trivial conservation laws. To any conservation law ω one can associate its characteristic ψ_ω which is a cosymmetry.

In what follows we shall mostly encounter *two-component* conservation laws for which the $(n-1)$-form ω in question has, in suitably chosen local coordinates, just two nonzero components, so its closeness condition can be written as

$$D_{x^{i_1}}(\rho) = D_{x^{i_2}}(\sigma)$$

on $\mathcal{E}$ for some specific indices i_1 and i_2.

A morphism of diffieties is a smooth map $\tau \colon \tilde{\mathcal{E}} \to \mathcal{E}$ which takes the Cartan distribution $\tilde{\mathcal{C}}$ on $\tilde{\mathcal{E}}$ to that on $\mathcal{E}$. A morphism τ is a (differential) covering if its differential maps the Cartan plane $\mathcal{C}_{\tilde{\theta}}$ to $\mathcal{C}_{\tau(\tilde{\theta})}$ isomorphically for any $\tilde{\theta} \in \tilde{\mathcal{E}}$. In other words, for any vector field Z on M the field $\tilde{\mathcal{C}}_Z$ projects to $\mathcal{C}_Z$. Thus, in local coordinates the total derivatives on $\tilde{\mathcal{E}}$ read

$$\tilde{D}_{x^i} = D_{x^i} + X_i, \qquad i = 1, \ldots, n, \tag{3}$$

where D_{x^i} are the total derivatives on $\mathcal{E}$, and

$$D_{x^i}(X_j) - D_{x^j}(X_i) + [X_i, X_j] = 0, \qquad 1 \le i < j \le n,$$

X_i being τ-vertical vector fields on $\tilde{\mathcal{E}}$. A covering $\tau\colon \tilde{\mathcal{E}} \to \mathcal{E}$ over a differentially connected diffiety is called irreducible if the covering diffiety $\tilde{\mathcal{E}}$ is differentially connected as well.

Symmetries, cosymmetries, conservation laws of the covering equation $\tilde{\mathcal{E}}$ are nonlocal symmetries, etc., of $\mathcal{E}$. Local objects depend on formal solutions of $\mathcal{E}$ and their partial derivatives; roughly speaking, nonlocal ones may depend on integrals of these solutions.

For example, the relations

$$\tilde{D}_x = D_x + \frac{wu}{2}, \qquad \tilde{D}_t = D_t + \frac{w}{2}\left(\frac{u^2}{2} + u_x\right) \tag{4}$$

define a covering of the Burgers equation $\mathcal{E} = \{u_t = uu_x + u_{xx}\}$ by the heat equation $\tilde{\mathcal{E}} = \{w_t = w_{xx}\}$.

Thus, the nonlocal variable w is related to u by the formulas

$$w_x = \frac{wu}{2}, \quad w_t = \frac{w}{2}\left(\frac{u^2}{2} + u_x\right). \tag{5}$$

Note that system (5) is compatible by virtue of the Burgers equation.

The form $\omega = w\,dx + w_x\,dt$ is a local conservation law of $\tilde{\mathcal{E}}$, and its pullback to $\mathcal{E}$ gives a nonlocal conservation law for $\mathcal{E}$. The corresponding nonlocal conserved density on $\mathcal{E}$, i.e., w, defined by (5), can be informally thought of as $\int \exp(u/2)\,dx$.

It should be also pointed out that the (isospectral) Lax pairs in this setting are just linear coverings involving an essential parameter, cf. e.g. [28] and references therein.

For example, an integrable 6D second-order PDE discovered in [S17],

$$u_s u_{zt} - u_z u_{st} - u_s u_{xy} + u_y u_{sx} - u_y u_{rz} + u_z u_{ry} = 0, \tag{6}$$

has a Lax pair given by [S17]

$$\chi_z - \frac{u_z}{u_s}\chi_s - \lambda\chi_x + \lambda\frac{u_z}{u_s}\chi_r = 0, \quad \chi_y - \frac{u_y}{u_s}\chi_s - \lambda\chi_t + \lambda\frac{u_y}{u_s}\chi_r = 0, \tag{7}$$

and (7) quite obviously defines a covering over (6) that involves an essential parameter λ.

Note that any $\mathcal{C}$-differential operator Δ on $\mathcal{E}$ can be lifted to a $\mathcal{C}$-differential operator $\tilde{\Delta}$ on $\tilde{\mathcal{E}}$ using equations (3). In particular, this can be done with the linearization operator $\ell_{\mathcal{E}}$ and its adjoint. Solutions of the equations

$$\tilde{\ell}_{\mathcal{E}}(\phi) = 0, \qquad \tilde{\ell}^*_{\mathcal{E}}(\psi) = 0$$

are called nonlocal *shadows* of symmetries and cosymmetries, respectively.

For the rest of this section consider, following mostly [35, 39], an evolution system in two independent variables, which we denote by x and t, and N dependent variables u^A:

$$\boldsymbol{u}_t = \boldsymbol{F}(x, t, \boldsymbol{u}, \boldsymbol{u}_1, \dots, \boldsymbol{u}_n), \tag{8}$$

where $\boldsymbol{u}_j = \partial^j \boldsymbol{u}/\partial x^j$ with the understanding that $\boldsymbol{u}_0 = \boldsymbol{u}$, and $\boldsymbol{F}$ is a smooth function of its arguments.

Let $\mathcal{S}_{\boldsymbol{F}}$ denote the diffiety associated with (8). It is immediate that we can take $x, t, \boldsymbol{u}, \boldsymbol{u}_1, \boldsymbol{u}_2, \dots$ for local coordinates on $\mathcal{S}_{\boldsymbol{F}}$. This choice greatly simplifies many of the above definitions, and in what follows we shall stick to it whenever we deal with a system of the form (8).

First of all, a local function is then just a smooth function of $x, t, \boldsymbol{u}$ and finitely many $\boldsymbol{u}_j$. We shall denote the algebra of such functions by $\mathcal{A}$.

The total derivatives on $\mathcal{S}_{\boldsymbol{F}}$ then read

$$D_x = \partial/\partial x + \sum_{i=0}^{\infty}\sum_{A=1}^{N} u^A_{i+1}\partial/\partial u^A_i, \quad D_t = \partial/\partial t + \sum_{i=0}^{\infty}\sum_{A=1}^{N} D^i(F^A)\partial/\partial u^A_i,$$

where u^A_i stand for $\partial^i u^A/\partial x^i$.

Next, a symmetry for (8) is identified with a local vector function $\boldsymbol{G}$ that satisfies

$$D_t(\boldsymbol{G}) - \boldsymbol{F}_*(\boldsymbol{G}) = 0, \tag{9}$$

where $\boldsymbol{F}_*$ is a linearization of $\boldsymbol{F}$, i.e., a $N \times N$-matrix-valued $\mathcal{C}$-differential operator with the entries

$$(\boldsymbol{F}_*)^A_B = \sum_{j=0}^{n} \partial F^A/\partial u^B_j D^j_x.$$

Likewise, a cosymmetry $\boldsymbol{\gamma}$ for (8) by definition must satisfy

$$D_t(\boldsymbol{\gamma}) + \boldsymbol{F}^\dagger_*(\boldsymbol{\gamma}) = 0, \tag{10}$$

where $\boldsymbol{F}^\dagger_*$ is the formal adjoint of $\boldsymbol{F}_*$.

In turn, without loss of generality we can assume that a conservation law for (8) is a horizontal differential one-form $\rho dx + \sigma dt$, where $\rho, \sigma \in \mathcal{A}$, such that

$$D_t(\rho) = D_x(\sigma); \tag{11}$$

ρ is called the density and σ the flux.

The characteristic of such a conservation law has the form $\delta\rho/\delta\boldsymbol{u}$, where

$$\delta/\delta\boldsymbol{u} = \sum_{i=0}^{\infty} (-D_x)^i \circ \partial/\partial\boldsymbol{u}_i$$

is the operator of variational derivative.

A conservation law $\rho dx + \sigma dt$ is said to be trivial if there is a $\zeta \in \mathcal{A}$ such that $\rho = D_x\zeta$ and $\sigma = D_t\zeta$.

3 Symmetries and conservation laws for evolution systems

In this section we present some results on symmetries and local conservation laws for certain evolution systems of the form (8).

It is well known, cf. e.g. [28, 39], that finding all generalized symmetries and/or all local conservation laws for a given system (8) is, in general, quite challenging and has been done for rather few systems to date.

3.1 Generalized symmetries

The search for generalized symmetries can often be greatly aided by the formal symmetry approach, see [35] and references therein, a modification of which allowing for explicit time dependence in the equation under study as well as its symmetries, can be found in [SV].

Using this enabled us to obtain inter alia the complete description of generalized symmetries for the following equation [23, 24] arising in the study of shallow water waves:

$$\begin{aligned} u_t \;&=\; -u_x - \frac{3}{2}auu_x - \frac{1}{6}bu_{xxx} + \frac{3}{8}a^2u^2u_x - \frac{23}{24}abu_xu_{xx} - \frac{5}{12}abuu_{xxx} \\ &+\frac{d}{2}(h'u + hu_x) + \frac{1}{4}bd(-h'''u - h''u_x + h'u_{xx} + hu_{xxx}) - \frac{19}{360}b^2u_{xxxxx}, \end{aligned} \tag{12}$$

where a, b, and d are constants, $h = h(x)$ is a smooth function of the primes indicate x-derivatives of h.

Theorem 1 ([SV]) *If $a \neq 0$, $b \neq 0$, and $d \neq 0$ then all generalized symmetries of (12) are equivalent to the Lie point ones:*

If $a \neq 0$, $b \neq 0$, $d \neq 0$, and $h' \not\equiv 0$, then the only generalized symmetry of (12) is the one with the characteristic equal to F; this symmetry corresponds to the Lie point symmetry $\partial/\partial t$, i.e., the time translation.

If $a \neq 0$, $b \neq 0$, $d \neq 0$ and $h \equiv \mathrm{const}$, then, in addition to the time translation, we have a symmetry with the characteristic u_x, which corresponds to the Lie point symmetry $\partial/\partial x$, i.e., the space translation.

Moreover, if $a \neq 0$, $b \neq 0$, $d \neq 0$, $h \equiv \text{const}$, and $hd = 4$, then in addition to the space and time translations equation (12) admits a symmetry with the characteristic $5tF + (x + 2t)u_x + 2u - 4/a$, which corresponds to a Lie point symmetry $5t\partial/\partial t + (x + 2t)\partial/\partial x + (4/a - 2u)\partial/\partial u$.

As an aside note that for the conservation laws and cosymmetries, using the formal conservation law technique one can prove the following

Theorem 2 ([SV]) *If $a \neq 0$, $b \neq 0$, and $d \neq 0$, then (12) has, modulo trivial conservation laws, just one local conservation law $\rho dx + \sigma dt$ with*

$$\begin{aligned} \rho &= u, \\ \sigma &= u - \frac{3}{4}au^2 - \frac{1}{6}bu_{xx} + \frac{1}{8}a^2u^3 - \frac{13}{48}abu_x^2 - \frac{5}{12}abuu_{xx} \\ &\quad + \frac{d}{2}hu + \frac{1}{4}bd(-h''u + hu_{xx}) - \frac{19}{360}b^2u_{xxxx} \,. \end{aligned} \tag{13}$$

associated to the only local cosymmetry $\gamma = 1$ of (12).

For another example, below is a complete description of generalized symmetries for the so-called Bakirov system [6]. This strengthens the earlier beautiful result of [8] stating that the system in question has just one genuinely generalized symmetry independent of x and t, completing the disprovement of a longstanding belief that existence of one genuinely generalized symmetry should imply existence of infinitely many.

Theorem 3 ([S01]) *Any generalized symmetry of the Bakirov system*

$$u_t = u_4 + v^2, \quad v_t = v_4/5, \tag{14}$$

where now $\boldsymbol{u} = (u, v)^T$, is a linear combination of symmetries with the characteristics from the following list:

$$\begin{aligned} \boldsymbol{Q}_1 &= (u_1, v_1)^T, & \boldsymbol{Q}_2 &= (u_4 + v^2, v_4/5)^T, \\ \boldsymbol{Q}_3 &= (2u, v)^T, & \boldsymbol{Q}_4 &= (4t(u_4 + v^2) + xu_1, 4tv_4/5 + xv_1 + 2v)^T, \\ \boldsymbol{R}_\alpha &= (\alpha(x,t), 0)^T, & \boldsymbol{Q}_5 &= (u_6 + 5(5vv_2 + 4v_1^2)/11, v_6)^T, \end{aligned}$$

where $\alpha(x,t)$ is any smooth solution of the linear equation $\partial\alpha/\partial t = \partial^4\alpha/\partial x^4$.

Thus, the only genuinely generalized (i.e., not equivalent to a Lie point one) symmetry for the Bakirov system is $\boldsymbol{Q}_5$.

3.2 Conservation laws for linear evolution equations

In this subsection we shall address the special case of (8) when $N = 1$, so $\boldsymbol{u} = u$, and $\boldsymbol{F} = F$ is linear in all u_j:

$$u_t = \sum_{i=0}^{k} a^i(t, x) u_i. \tag{15}$$

We tacitly assume here that $a_k \neq 0$.

Without loss of generality we shall assume conservation laws to be of the form (11) and consider them modulo trivial ones.

For equations (15) quite a lot can be said about their conservation laws and cosymmetries. In particular, the following results hold true:

Theorem 4 ([PS]) *For any linear evolution equation (15) of even order $k \geq 2$, all its cosymmetries depend only on x and t, and the space of all cosymmetries is isomorphic to the space of (smooth) solutions of the associated adjoint equation*

$$\frac{\partial f}{\partial t} + \sum_{i=0}^{k} (-1)^i \frac{\partial^i}{\partial x^i} \left(a^i(t, x) f \right) = 0. \tag{16}$$

where $f = f(x, t)$.

Corollary 1 ([PS]) *For any linear evolution equation (15) of even order $k \geq 2$ its space of conservation laws is exhausted by those with densities linear in u and is isomorphic to the space of (smooth) solutions of the associated adjoint equation (16).*

Theorem 5 ([PS]) *For any linear evolution equation (15) of odd order $k \geq 3$, all its cosymmetries are affine in the totality of variables $u_0, u_1, u_2, \dots$.*

An immediate consequence of the preceding theorem is the following

Theorem 6 ([PS]) *For any linear evolution equation (15) of odd order $k \geq 3$, the space of its conservation laws is spanned by those with densities that are at most quadratic in u_j, $j = 0, 1, 2, \ldots$.*

3.3 Hamiltonian structures and all that

We begin with a brief review of Hamiltonian structures and related matters following mostly [9, 39].

Consider first a finite-dimensional smooth manifold, say V. Then a bivector (i.e., a (2,0)-tensor field) P on V is called a *Poisson structure*, or a *Hamiltonian structure*, if its Schouten bracket with itself vanishes or, equivalently, the associated bracket on $C^\infty(V)$, given by

$$\{f, g\} =< df, Pdg >$$

for any $f, g \in C^\infty(V)$ is skewsymmetric and satisfies the Jacobi identity (and thus is a Poisson bracket on V. Here and below we shall mostly interpret a Poisson structure as a linear operator sending one-forms on V into vector fields on V.

Two Poisson structures P and $\tilde{P}$ are *compatible* if any linear combination of the two again is a Poisson structure.

A Poisson structure is *nondegenerate* if $\ker P = 0$.

A vector field X on V (and the associated dynamical system for its integral curves) is said to be *Hamiltonian* w.r.t. a Poisson structure P with a Hamiltonian $H \in C^\infty(V)$ if X can be written as $X = PdH$.

Likewise, X is *bihamiltonian* w.r.t. a pair of compatible Poisson structures P and $\tilde{P}$ if there exist $H, \tilde{H} \in C^\infty(V)$ such that X can be written as $X = PdH = \tilde{P}d\tilde{H}$; see the seminal paper [32] and e.g. [9, 39] for further details.

What makes (bi)hamiltonian systems particularly interesting is the well-known fact that under a number of not too restrictive assumptions X being bihamiltonian implies integrability by quadratures of the dynamical system for the integral curves of X, see e.g. [9, 39] and references therein for details.

This naturally leads to the following question: given a Poisson structure P, how can one describe *all* Poisson structures compatible with P?

For nondegenerate P, there is a concise characterization of such structures:

Theorem 7 ([S04]) *Suppose that $H^2(V) = 0$, i.e., the second de Rham cohomology of V is trivial, and let P be a nondegenerate Poisson structure on V. Then a bivector $\tilde{P}$ on V is a Poisson structure compatible with P if and only if there exist vector fields τ and $\tilde{\tau}$ on V such that*

$$\tilde{P} = L_\tau(P) \tag{17}$$

and

$$L_\tau^2(P) = L_{\tilde{\tau}}(P). \tag{18}$$

Here and below L_X stands for the Lie derivative along the vector field X.

It is clearly easier to look for pairs of vector fields τ and $\tilde{\tau}$ that satisfy (18) for a given nondegenerate P than to look for all Poisson structures compatible with P using just the definition of compatibility.

Moreover, the idea of the above theorem can be readily generalized to certain classes of Poisson structures associated to (evolutionary) partial differential systems, when the remark from the previous paragraph is particularly relevant, see Theorem 8 below for one such generalization. Note that in this context Poisson structures are more often referred to as Hamiltonian structures, see e.g. [12, 39].

Let us describe, mostly following [12, 39], the simplest yet still quite interesting setup for this case. Like as in our earlier discussion of evolution systems in two independent variables, let a local function now be just a smooth function of $x, t, \boldsymbol{u}$ and finitely many $\boldsymbol{u}_j$, and denote the algebra of local functions by $\boldsymbol{\mathcal{A}}$.

Let $\mathcal{V}$ denote the Lie algebra of evolutionary vector fields of the form

$$\boldsymbol{v}_\phi = \sum_{A=1}^{N} \sum_{j=0}^{\infty} D_x^j(\phi^A)\partial/\partial u_j^A$$

where $\phi^A \in \mathcal{A}$ (recall that ϕ is called a characteristic of such a vector field, cf. e.g. [39]). As an evolutionary vector field $\boldsymbol{v}_\phi$ is uniquely determined by its characteristic ϕ, as a vector space $\mathcal{V} \simeq \mathcal{A}^N$. We shall implicitly make use of this below.

The dual $\mathcal{V}^*$ of $\mathcal{V}$ in our setting is the space of 1-forms of the form

$$\omega = \sum_{A=1}^{N} \omega_A du^A$$

with $\omega_A \in \mathcal{A}$.

The pairing is defined as follows:

$$< \omega, \boldsymbol{v}_\phi >= \int dx \sum_{A=1}^{N} \phi^A \omega_A$$

with integral $\int dx$ understood in the spirit of formal calculus of variations [12, 39].

We also need the space $\mathcal{L}$ of functionals of the form $\mathcal{H} = \int h dx$, where $h \in \mathcal{A}$. For such a functional its variation is defined as

$$\delta\mathcal{H} = \delta h / \delta \boldsymbol{u}.$$

A linear $\mathcal{C}$-differential operator $P : \mathcal{V}^* \to \mathcal{V}$ is a *Hamiltonian operator* if the associated bracket defined by the formula

$$\{\mathcal{G}, \mathcal{H}\} =< \delta\mathcal{G}, P\delta\mathcal{H} > \quad \forall \mathcal{G}, \mathcal{H} \in \mathcal{L}$$

is a Poisson bracket on $\mathcal{L}$, i.e., it is skewsymmetric and satisfies the Jacobi identity.

In local coordinates such an operator has the form

$$P = \sum_{i=0}^{s} p_i D_x^i$$

where p_i are $N \times N$ matrices whose entries belong to $\mathcal{A}$.

Note that the requirement of being $\mathcal{C}$-differential can be relaxed to allow for certain nonlocalities, cf. e.g. [12, 17, 27, 28] and references therein for details.

Thus, we can think of P as of an operator that sends an element of $\mathcal{V}^*$ into a characteristic of an evolutionary vector field.

With this in mind, an evolution system (8) is *Hamiltonian* with respect a Hamiltonian operator P with a Hamiltonian $\mathcal{H} \in \mathcal{L}$ if the right-hand side of (8) can be written as $\boldsymbol{F} = P\delta\mathcal{H}$.

Note that for a system of the form (8) a functional $\mathcal{H} = \int h dx \in \mathcal{L}$ is said to be an integral of motion if h is a conserved density, that is, there is a $\sigma \in \mathcal{A}$ such that $D_t(h) = D_x(\sigma)$ and thus $hdx + \sigma dt$ is a conservation law for the system under study.

The definition of bihamiltonian system then mimics that of a bihamiltonian vector field above, with X replaced by $\boldsymbol{F}$ and d by δ, cf. e.g. [9, 12, 39].

Again, just as in the case of finite-dimensional integrable dynamical systems, if a system (8) is bihamiltonian, then, under certain technical assumptions it admits, see e.g. [9, 12, 39] and references therein, infinitely many commuting symmetries and integrals of motion that commute with respect to both Poisson brackets associated with the relevant Hamiltonian operators.

This makes finding bihamiltonian representation for a given partial differential system, which in many cases is quite a nontrivial task, an important step in establishing integrability of the latter.

Consider the operators P from $\mathcal{V}^*$ to $\mathcal{V}$, that can be written as $P = ||P^{AB}||$ where

$$P^{AB} = g^{AB} D_x + \sum_{C=1}^{N} f_C^{AB} u_1^C, \tag{19}$$

and g^{AB} and f_C^{AB} depend only on $\boldsymbol{u}$.

The seminal result of Dubrovin and Novikov [15] is that if g^{AB} defines a nondegenerate (pseudo-)Riemannian metric g then the following holds: P defined by (19) is a Hamiltonian operator if and only if g is flat and we have

$f_C^{AB} = -\sum_{D=1}^N g^{AD}\Gamma_{DC}^B$ where the Γ's are components of the Levi–Civita connection for g. Such Hamiltonian operators are known as Hamiltonian operators of Dubrovin–Novikov type. Note that if g is degenerate, the conditions for (19) to define a Hamiltonian operator are significantly more complicated, see e.g. [43] for details.

There is the following analog of Theorem 7 for this case:

Theorem 8 ([S04]) *Let P be a Hamiltonian operator of Dubrovin–Novikov type, i.e. of the form (19) with nondegenerate g.*

Consider another operator $\tilde{P} : \mathcal{V}^* \to \mathcal{V}$ from the class (19), that is, of the form $\tilde{P} = ||\tilde{P}^{AB}||$, where

$$\tilde{P}^{AB} = \tilde{g}^{AB} D_x + \sum_{C=1}^N \tilde{f}_C^{AB} u_1^C, \tag{20}$$

and $\tilde{g}^{AB}$ and $\tilde{f}_C^{AB}$ depend only on $\boldsymbol{u}$.

Then any such $\tilde{P}$ is a Hamiltonian operator compatible with P if and only if there exist, in general only locally defined, $\tau \in \mathcal{A}^N$ and $\tilde{\tau} \in \mathcal{A}^N$ that depend on $\boldsymbol{u}$ alone, such that the following conditions hold:
i) $\tilde{P}$ can be written as $\tilde{P} = L_{\boldsymbol{v}_\tau}(P)$;
ii) $L_{\boldsymbol{v}_\tau}^2(P) = L_{\boldsymbol{v}_{\tilde{\tau}}}(P)$.

We stress that the nondegeneracy of g is assumed in the above theorem but that of $\tilde{g}$ is not.

The term 'locally defined' here means that τ and $\tilde{\tau}$ in general can be defined only on certain open domains but not necessarily for all $\boldsymbol{u}$. In the language of Section 2 τ and $\tilde{\tau}$ can, roughly speaking, be thought of as local (rather than global) vector fields on E. For the details on the definition of Lie derivative for Hamiltonian operators see e.g. [9, 12].

Now turn to the problem of finding bihamiltonian representations for specific evolutionary systems (8). There is no general recipe for finding those, and sometimes this can be quite a challenge, as illustrated by the following result.

Theorem 9 ([S05]) *The system found in [22],*

$$\begin{aligned} u_t &= 4u_{xxx} - v_{xxx} - 12uu_x + vu_x + 2uv_x, \\ v_t &= 9u_{xxx} - 2v_{xxx} - 12vu_x - 6uv_x + 4vv_x, \end{aligned}$$

where now $\boldsymbol{u} = (u, v)^T$, is bihamiltonian:

$$\boldsymbol{u}_t = P_1 \delta \mathcal{H}_0 = P_0 \delta \mathcal{H}_1, \tag{21}$$

where $\mathcal{H}_0 = \int dx(v/2 - 3u)$, $\mathcal{H}_1 = \int dx(2u^2 - uv + v^2/9)$, and P_0 and P_1 are compatible Hamiltonian operators of the form

$$P_0 = \begin{pmatrix} D_x^3 - 2uD_x - u_x & 0 \\ 0 & -9D_x^3 + 12vD_x + 6v_x \end{pmatrix}, P_1 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

where

$$\begin{aligned} a_{11} &= D_x^5 - 4uD_x^3 - 6u_xD_x^2 + 4(u^2 - u_{xx})D_x - u_{xxx} + 4uu_x - u_xD_x^{-1} \circ u_x, \\ a_{12} &= 2D_x^5 - (2u + 3v)D_x^3 + 4(u_x - 2v_x)D_x^2 + (6u_{xx} - 7v_{xx} - 4u^2 + 6uv)D_x \\ &\quad +2u_{xxx} - 2v_{xxx} - 6uu_x + 3vu_x + 4uv_x - u_xD_x^{-1} \circ v_x, \\ a_{21} &= 2D_x^5 - (2u + 3v)D_x^3 - (10u_x + v_x)D_x^2 + (-4u^2 + 6uv - 8u_{xx})D_x \\ &\quad -2u_{xxx} - 2uu_x + 3vu_x + 2uv_x - v_xD_x^{-1} \circ u_x, \\ a_{22} &= 3D_x^5 + (18u - 12v)D_x^3 + (27u_x - 12v_x)D_x^2 \\ &\quad +(21u_{xx} - 14v_{xx} - 12u^2 - 12uv + 9v^2)D_x \\ &\quad +6u_{xxx} - 4v_{xxx} - 12uu_x - 6vu_x - 6uv_x + 9vv_x - v_xD_x^{-1} \circ v_x. \end{aligned}$$

Define the quantities $\boldsymbol{Q}_j$ and $\mathcal{H}_j = \int h_j dx$ recursively by the formula $\boldsymbol{Q}_j = P_1 \delta \mathcal{H}_j = P_0 \delta \mathcal{H}_{j+1}$, $j = 0, 1, 2, \ldots$. Then h_j, $j = 2, 3, \ldots$, are local functions that can be chosen to be independent of x and t, and $\boldsymbol{v}_{\boldsymbol{Q}_j}$ are local commuting generalized symmetries for (21) for all $j = 1, 2, \ldots$.

Moreover, the evolution systems $\boldsymbol{u}_{t_j} = \boldsymbol{Q}_j$ are bihamiltonian with respect to P_1 and P_0 by construction, and $\mathcal{H}_j = \int h_j dx$ are in involution with respect to the Poisson brackets associated with P_0 and P_1 for all $j = 0, 1, 2 \ldots$, so $\mathcal{H}_j$ are joint integrals of motion for all evolution systems $\boldsymbol{u}_{t_k} = \boldsymbol{Q}_k$, $k = 0, 1, 2, \ldots$.

Another system whose bihamiltonian represenattion was elusive for quite a long time is presented below.

Theorem 10 ([SD]) *The Sasa–Satsuma [40] system*

$$p_t = p_{xxx} + 9pqp_x + 3p^2 q_x, \quad q_t = q_{xxx} + 9pqq_x + 3q^2 p_x \tag{22}$$

is a bihamiltonian system with a pair of compatible Hamiltonian operators

$$P = \begin{pmatrix} -pD_x^{-1} \circ p & D_x + pD_x^{-1} \circ q \\ D_x + qD_x^{-1} \circ p & -qD_x^{-1} \circ q \end{pmatrix},$$

and $\tilde{P} = P \circ J \circ P$. *Here* J *is a symplectic operator of the form*

$$J = \begin{pmatrix} 3qD_x^{-1} \circ q & D_x + 2pD_x^{-1} \circ q + 3qD_x^{-1} \circ p \\ D_x + 2qD_x^{-1} \circ p + 3pD_x^{-1} \circ q & 3pD_x^{-1} \circ p \end{pmatrix}.$$

The bihamiltonian representation for (22) reads

$$\boldsymbol{u}_t = P\delta\mathcal{H}_0 = \tilde{P}\delta\mathcal{H}_1$$

where now $\boldsymbol{u} = (p,q)^T$, $\mathcal{H}_0 = \int dx(2p^2q^2 - p_x q_x)$, $\mathcal{H}_1 = \int dx\ pq$.

In this case establishing locality of $\mathcal{H}_j$ and $\boldsymbol{Q}_j$ defined through the relations $\boldsymbol{Q}_j = \tilde{P}\delta\mathcal{H}_j = P\delta\mathcal{H}_{j+1}$ mimicking those in Theorem 9, and locality of $\boldsymbol{Q}_j$ is more difficult than e.g. for (21), and was done in the later work by Wang [45] who has also fixed some typos in J; J in the above theorem already incorporate her corrections.

4 Nonlocal objects related to integrable systems

4.1 Commuting nonlocal symmetries

Consider the four-dimensional Martínez Alonso–Shabat equation

$$u_{ty} = u_z\, u_{xy} - u_y\, u_{xz} \tag{23}$$

introduced in [2]. It has [36] a covering defined by system

$$q_y = \lambda\, u_y\, q_x, \qquad q_z = \lambda\, (u_z\, q_x - q_t) \tag{24}$$

with a non-removable parameter λ (which means that (24) defines a Lax pair for (23)), and a recursion operator, and is therefore integrable.

Moreover, there is another Lax pair for (23)

$$w_y = \lambda\, (u_y\, w_x - u_{xy}\, w), \qquad w_z = \lambda(u_z\, w_x - w_t - u_{xz}\, w) \tag{25}$$

that defines another covering over (23) with an important property: if w satisfies (25) then it is a nonlocal symmetry shadow for (23).

We have the following important observation.

Consider a copy of the covering (25) with the parameter μ:

$$\tilde{w}_y = \mu\, (u_y\, \tilde{w}_x - u_{xy}\, \tilde{w}), \qquad w_z = \mu(u_z\, \tilde{w}_x - \tilde{w}_t - u_{xz}\, \tilde{w}) \tag{26}$$

Proposition 1 ([MS]) *The system*

$$u_\tau = \tilde{w}, \quad w_\tau = \frac{\lambda\, \mu}{\mu - \lambda}\, (\tilde{w}\, w_x - w\, \tilde{w}_x)$$

is compatible with (23) and (25), i.e.,

$$\tilde{w}\frac{\partial}{\partial u} + \frac{\lambda\, \mu}{\mu - \lambda}\, (\tilde{w}\, w_x - w\, \tilde{w}_x)\, \frac{\partial}{\partial w}, \tag{27}$$

is a symmetry for (23)+(25).

Substituting into (25) a formal expansion $w = \sum_{i=0}^{\infty} w_i \lambda^i$ gives rise to another covering over (23) generated by the system

$$\begin{aligned} (w_i)_y &= u_y\,(w_{i-1})_x - u_{xy}\,w_{i-1}, \\ (w_i)_z &= u_z\,(w_{i-1})_x - (w_{i-1})_t - u_{xz}\,w_{i-1} \end{aligned} \tag{28}$$

for $i = 1, 2, \dots$; w_0 is an arbitrary smooth function of t and x.

More precisely, (28) defines an infinite-dimensional covering, say $\mathcal{W}$, over (23) with the nonlocal variables w_{iab}, and when spelled out in full this covering is given by

$$(w_{iab})_x = w_{i,a+1,b}, \quad (w_{iab})_t = w_{i,a,b+1}, \quad a, b = 0, 1, 2, \dots \tag{29}$$

with $w_{i00} \equiv w_i$, along with (28), and differential consequences of both (28) and (29).

It is readily checked that w_i are nonlocal shadows for (23), and, most importantly, using the above proposition and performing formal expansions w.r.t. the parameters involved, it can be shown that these shadows can be promoted to full-fledged nonlocal symmetries of (23) in the covering $\mathcal{W}$, and these symmetries commute:

Theorem 11 ([MS]) *The infinite prolongations of the vector fields*

$$Q_i = w_i \frac{\partial}{\partial u} + \sum_{j=1}^{\infty} \sum_{k=0}^{i-1} (w_{i+j-k-1}\,(w_k)_x - w_k\,(w_{i+j-k-1})_x)\,\frac{\partial}{\partial w_j}, \tag{30}$$

where $i \in \mathbb{N}$, upon restriction to (23) and (28) form an infinite series of commuting nonlocal symmetries for equation (23) in the covering (28).

It is important to stress that finding a nonlocal symmetry similar to (27) for a system consisting of a nonlinear partial differential system and its Lax pair, like (23)+(25), is fairly straightforward, so the likes of Proposition 1 and hence of Theorem 11 can be proved for many other integrable multidimensional systems with isospectral Lax pairs, enabling one to obtain explicit form of the associated hierarchies of nonlocal commuting symmetries. Such hierarchies are intimately related to integrability and are rather difficult to construct by other means as the recursion operators for systems in more than two independent variables typically produce nonlocal shadows rather than full-fledged nonlocal symmetries, cf. e.g. [28, S17] and references therein.

4.2 Recursion operators

Recall that, informally, a recursion operator for a given system is an operator that maps a symmetry of the system under study into (another) symmetry, cf. the seminal paper [38] and e.g. [9, 16, 39]. However, the recursion operators for partial differential systems often involve nonlocalities, and it turns out that a proper way to handle those is to think of a recursion operator as of a Bäcklund auto-transformation for the linearized version of the system under study, see [21, 34, 28] and references therein for details. Note that for certain systems in more than two independent variables there exist recursion operators of different type, namely bilocal ones, see e.g. [9, 19] and references therein, but this is beyond our scope here.

As it was already pointed out a bit earlier, existence of infinite hierarchies of symmetries is an important feature of integrable partial differential systems, and the recursion operators provide a highly useful tool for generating such hierarchies, cf. e.g. [28, 39].

Consider a partial differential system

$$F_I(\vec{x}, [\boldsymbol{u}]) = 0, \quad I = 1, \dots, m, \tag{31}$$

denote by $\mathcal{F}$ the associated diffiety, and introduce the following notation for operators in total derivatives:

$$\begin{aligned} A_i &= A_i^0 + \sum_{j=1}^{n} A_i^j D_{x^j}, \quad B_i = B_i^0 + \sum_{j=1}^{n} B_i^j D_{x^j}, \quad i = 1, 2, \\ L &= L^0 + \sum_{k=1}^{n} L^k D_{x^k}, \quad M = M^0 + \sum_{k=1}^{n} M^k D_{x^k}. \end{aligned} \tag{32}$$

Here $A_i^j = A_i^j(\vec{x}, [\boldsymbol{u}])$ and $B_i^j = B_i^j(\vec{x}, [\boldsymbol{u}])$ for $i = 1, 2$ and $j = 1, \ldots, n$ are scalar functions, $A_i^0 = A_i^0(\vec{x}, [\boldsymbol{u}])$ and $B_i^0 = B_i^0(\vec{x}, [\boldsymbol{u}])$ for $i = 1, 2$ are $N \times N$ matrices, $L^k = L^k(\vec{x}, [\boldsymbol{u}])$ for $k = 0, \ldots, n$ are $N \times m$ matrices, and $M^k = M^k(\vec{x}, [\boldsymbol{u}])$ for $k = 0, \ldots, n$ are $m \times N$ matrices.

Theorem 12 ([S17]) *Suppose that for a system (31) there exist the operators A_i, B_i, L, M of the form (32) such that*

i) $[A_1, A_2] = 0,$ (33)

ii) $[B_1, B_2] = 0,$ (34)

iii) $(A_1 B_2 - A_2 B_1) = L \circ \ell_{\mathcal{F}},$ (35)

iv) $\ell_{\mathcal{F}} = M \circ (B_1 A_2 - B_2 A_1),$ (36)

v) *there exist $p, q \in \{1, \ldots, n\}$, $p \neq q$, such that we can express $D_{x^p}\tilde{\boldsymbol{U}}$ and $D_{x^q}\tilde{\boldsymbol{U}}$ from the relations*

$$A_i(\tilde{\boldsymbol{U}}) = B_i(\boldsymbol{U}), \quad i = 1, 2. \tag{37}$$

Then relations (37) define a recursion operator for (31), i.e., whenever $\boldsymbol{U}$ is a nonlocal symmetry shadow for (31), so is $\tilde{\boldsymbol{U}}$ defined by (37).

A natural source of A_i, B_i and L, M satisfying the conditions of Proposition 12 is provided by the Lax pairs for (31) of the form

$$\mathcal{L}_i \psi = 0, \quad i = 1, 2, \tag{38}$$

with $\mathcal{L}_i$ linear in λ such that ψ is a nonlocal symmetry shadow for (31). Then putting $\mathcal{L}_i = \lambda B_i - A_i$ or $\mathcal{L}_i = \lambda A_i - B_i$ gives us natural candidates for A_i and B_i which then should be checked against the conditions of Proposition 12, and, if the latter hold, yield a recursion operator for (31).

For example, consider the general heavenly equation [13, 41]

$$a u_{xy} u_{zt} + b u_{xz} u_{yt} + c u_{xt} u_{yz} = 0, \qquad a + b + c = 0, \tag{39}$$

where a, b, c are constants.

Proposition 2 ([S17]) *If U is a nonlocal symmetry shadow for (39) then*

$$\tilde{U}_x = \frac{u_{xz}U_t + cu_{xt}\tilde{U}_z - u_{zt}U_x}{cu_{zt}}, \quad \tilde{U}_y = -\frac{u_{yz}U_t - bu_{yt}\tilde{U}_z - u_{zt}U_y}{bu_{zt}} \tag{40}$$

defines another nonlocal symmetry shadow $\tilde{U}$ for (39), i.e., the relations (40) define a recursion operator for (39).

It is important to stress that the Lax pair with the operators $\mathcal{L}_i$ employed in the above construction does not have to be the *original* Lax pair of our system (31). In general, we should *custom tailor* the operators $\mathcal{L}_{1,2}$ constituting the Lax pair for the construction in question, so that the solutions of the associated linear problem (38) are shadows of nonlocal symmetries, i.e., satisfy the linearized version of our system, see [S17] for details.

The method in question, based on the above remarks and Theorem 12, is quite straightforward to apply and works for plenty of examples of multidimensional integrable systems with isospectral Lax pairs, and other authors have already found a number of new recursion operators using our approach, see e.g. [7, 29]. It should also be noted that our method, when applicable, requires significantly less computations than e.g. that of [37].

5 Integrable systems in four independent variables

Among integrable systems, those in four independent variables (4D) are of particular interest, as four is the dimension of our spacetim according to general relativity, so gaining a deeper understanding of such systems could be quite significant for possible applications, including those in physics.

Such systems were long believed to be quite scarce, and an efficient construction for such systems remained elusive. The overwhelming majority of previously known integrable 4D systems are dispersionless in the following sense.

A partial differential system is said (cf. e.g. [10, 18] and references therein) to be of *hydrodynamic type*, or *dispersionless*, if it can be written as a first-order homogeneous quasilinear system, that is,

$$A_1(\boldsymbol{u})\boldsymbol{u}_{x^1} + A_2(\boldsymbol{u})\boldsymbol{u}_{x^2} + \cdots + A_n(\boldsymbol{u})\boldsymbol{u}_{x^n} = 0; \tag{41}$$

A_i are $M \times N$ matrices, $M \geqslant N$, $\boldsymbol{u} \equiv (u^1, \ldots, u^N)^T$; $\vec{x} = (x^1, \ldots, x^n)^T$, $\boldsymbol{u} = \boldsymbol{u}(\vec{x})$. Such systems have many applications including e.g. fluid dynqamics, nonlinear optics, and general relativity, see for example [10, 16, 18, 46] and references therein.

In what follows we shall deal with dispersionless systems in four independent variables that will be denoted x, y, z, t, so the systems under study will read

$$A_1(\boldsymbol{u})\boldsymbol{u}_x + A_2(\boldsymbol{u})\boldsymbol{u}_y + A_3(\boldsymbol{u})\boldsymbol{u}_z + A_4(\boldsymbol{u})\boldsymbol{u}_t = 0, \tag{42}$$

For an $h = h(p, \boldsymbol{u})$ define an operator X_h as

$$X_h = h_p\partial_x + (ph_z - h_x)\partial_p + (h - ph_p)\partial_z \tag{43}$$

which formally looks exactly like the contact vector field with a contact hamiltonian h on a contact 3-manifold with local coordinates x, z, p and contact one-form $dz + pdx$, see [S18] for details.

A linear system

$$\chi_y = X_f(\chi), \quad \chi_t = X_g(\chi) \tag{44}$$

for $\chi = \chi(x, y, z, t, p)$ will be hereinafter referred to [S18] as a *linear contact Lax pair*. Here p is the so-called variable spectral parameter (recall that $\boldsymbol{u} = \boldsymbol{u}(x, y, z, t)$, so $\boldsymbol{u}_p \equiv 0$, $f = f(p, \boldsymbol{u})$, $g = g(p, \boldsymbol{u})$ are the *Lax functions* and $L = \partial_y - X_f$ and $M = \partial_t - X_g$ are the *Lax operators*.

Fix f and g in (44) and consider for a moment the associated *Lax equation*

$$[\partial_y - X_f, \partial_t - X_g] = 0, \tag{45}$$

expressing the compatibility condition for (44). This compatibility condition can be expressed in a more concise form that simplifies many computations.

Proposition 3 ([S18]) *The Lax equation (45) holds iff so does*

$$f_t - g_y + \{f, g\} = 0, \tag{46}$$

where $\{f, g\} = f_p g_x - g_p f_x - p\,(f_p g_z - g_p f_z) + f g_z - g f_z$ *is a (special case of) the so-called contact bracket.*

The Lax pairs (44) provide a new and natural 4D generalization of a well-known (see e.g. [10, 11, 16, 46] and references therein) 3D Lax pairs

$$\chi_y = \mathcal{X}_f(\chi), \quad \chi_t = \mathcal{X}_g(\chi), \tag{47}$$

where $\mathcal{X}_h = h_p \partial_x - h_x \partial_p$, since if $\boldsymbol{u}_z = 0$ and $\chi_z = 0$ then (44) boils down to (47). The class of integrable 3D systems with Lax pairs (47) is quite broad, see e.g. [10, 18, 33, 46], so it is natural to ask whether this holds true for the class of integrable 4D systems with linear contact Lax pairs (44).

The following result shows that this is indeed the case and there is infinitely many pairs (f, g) such that the systems for $\boldsymbol{u}$ with Lax pairs (44) are new genuinely 4D integrable nonlinear systems transformable into Cauchy–Kowalevski form.

Theorem 13 ([S18]) *Linear contact Lax pairs yield new integrable 4D systems that can be brought into Cauchy–Kowalevski form for the following pairs of f and g:*

1. $f = p^{k+1} + \sum_{i=0}^{k} u_i p^i, \; g = p^{l+1} + \dfrac{l}{k} u_l p^l + \sum_{j=0}^{l-1} v_j p^j$

with $\boldsymbol{u} = (u_0, \dots, u_k, v_0, \dots, v_{l-1})^T$*;*

2. $f = \sum_{i=1}^{k} \dfrac{a_i}{(p - u_i)}, \quad g = \sum_{j=1}^{l} \dfrac{b_j}{(p - v_j)}$

with $\boldsymbol{u} = (a_1, \dots, a_k, u_1, \dots, u_k, b_1, \dots, b_l, v_1, \dots, v_l)^T$*.*

Here $k, l = 1, 2, 3, \dots$ *are arbitrary natural numbers.*

Note that linear contact Lax pairs (44) belong to a broader class of nonisospectral[1] Lax pairs

$$\begin{aligned} \chi_y &= K_1(p, \boldsymbol{u})\chi_x + K_2(p, \boldsymbol{u})\chi_z + K_3(p, \boldsymbol{u})\chi_p, \\ \chi_t &= L_1(p, \boldsymbol{u})\chi_x + L_2(p, \boldsymbol{u})\chi_z + L_3(p, \boldsymbol{u})\chi_p, \end{aligned} \tag{48}$$

and hence are amenable to an appropriate version of the inverse scattering transform, cf. e.g. [33] and references therein, which paves the way to constructing explicit exact solutions of the nonlinear systems admitting such Lax pairs. For the discussion of geometric approach to the Lax pairs (48) see e.g. [11] and references therein.

Let us also mention the following

Proposition 4 ([S18]) *A system (42) admits a linear contact Lax pair of the form (44) if and only if it admits a nonlinear Lax pair for* $\psi = \psi(x, y, z, t)$ *of the form*

$$\psi_y = \psi_z f(\psi_x/\psi_z, \boldsymbol{u}), \quad \psi_t = \psi_z g(\psi_x/\psi_z, \boldsymbol{u}) \tag{49}$$

with the same functions f *and* g *as in (44).*

Note that (49) is nothing but a pair of nonstationary Hamilton–Jacobi equations. Such systems are a special case of multitime Hamilton–Jacobi systems that were intensively studied in a different context, cf. e.g. [30].

[1] Roughly speaking, nonisospectrality here refers to the fact that the Lax pairs in question involve the derivatives with respect to p; for nonisospectral Lax pairs in general see e.g. [10, 11, 16] and references therein.

For example, let $\boldsymbol{u} = (u, v, w, r)^T$, and $f = p^2 + wp + u$, $g = p^3 + 2wp^2 + rp + v$, i.e., $k = 1$, $l = 2$, $u_0 \equiv u, u_1 \equiv w, v_0 \equiv v, v_1 \equiv r$, in the first of two classes from Theorem 13.

The linear contact Lax pair $\chi_y = X_f(\chi)$, $\chi_t = X_g(\chi)$ then reads

$$\begin{aligned}\chi_y &= (2p + w)\chi_x + (-p^2 + u)\chi_z + (w_z p^2 + (u_z - w_x)p - u_x)\chi_p, \\ \chi_t &= (r + 4wp + 3p^2)\chi_x + (v - 2wp^2 - 2p^3)\chi_z \\ &\quad +(2w_z p^3 + (r_z - 2w_x)p^2 + (v_z - r_x)p - v_x)\chi_p,\end{aligned} \tag{50}$$

while the nonlinear Lax pair as per Proposition 4 has the form

$$\psi_y = \psi_z \left(\left(\frac{\psi_x}{\psi_z} \right)^2 + w\frac{\psi_x}{\psi_z} + u \right), \quad \psi_t = \psi_z \left(\left(\frac{\psi_x}{\psi_z} \right)^3 + 2w \left(\frac{\psi_x}{\psi_z} \right)^2 + r\frac{\psi_x}{\psi_z} + v \right).$$

The associated integrable system for $\boldsymbol{u}$ reads

$$\begin{aligned}&u_t - vu_z - ru_x + uv_z + wv_x - v_y = 0, \\ &2u_z + w_x + 2ww_z - r_z = 0, \\ &2r_x - 3u_x - 2w_y + 2wu_z - v_z - 2ww_x + 2uw_z = 0, \\ &w_t - r_y + 2v_x - 4wu_x + wr_x - rw_x - vw_z + ur_z = 0.\end{aligned} \tag{51}$$

Note [S18] that the above system provides a novel integrable 4D generalization for the well-known integrable dispersionless Kadomtsev–Petviashvili equation, see e.g. [16, 46] and references therein for details on this equation.

Let us also point out that using a formal expansion of χ in p enables one to find an infinite hierarchy of nonlocal conservation laws for (51) using (50), see [S18] for details, and the same can be done for many other integrable systems with linear contact Lax pairs (44).

It should be pointed out that there are integrable 4D systems with contact Lax pairs whose Lax functions are not rational. Below we give an example involving algebraic Lax functions which, to the best of our knowledge, is the first example of an integrable 4D system with a nonisospectral Lax pair which is not rational in the spectral parameter.

Consider [S19] the following 4D evolutionary system for $\boldsymbol{u} = (a, b, r, s, u, v, w)^{\mathrm{T}}$:

$$
\begin{aligned}
a_t &= \frac{1}{r^2 - 2rsa + 2s^2b}\Big((4w(ra - sb) - vr)a_x + ra_y \\
&\quad +\big(2w\,(2a(ra - sb) - rb) - ur\big)a_z \\
&\quad +(vs - 2wr)b_x - sb_y + (2w(sb - ra) + us)b_z \\
&\quad +(r - sa)u_x + (ra - 2sb)u_z + (2sb - ra)v_x \\
&\quad +2(a(sb - ra) + rb)v_z + 2(a(ar - sb) - rb)w_x \\
&\quad +2\left(2a^2(ar - sb) - 3rab + 2sb^2\right) w_z\Big), \\
b_t &= \frac{1}{r^2 - 2rsa + 2s^2b}\Big(2(2wr - vs)ba_x + 2sba_y \\
&\quad +2(2w(ra - sb) - us)ba_z \\
&\quad +(2s(va - 2wb) - vr)b_x + (r - 2sa)b_y \\
&\quad +(2(usa - wrb) - ur)b_z \\
&\quad +(2s(b - a^2) + ra)u_x + 2(r - sa)bu_z \\
&\quad -2(r - sa)bv_x - 2(ra - 2sb)bv_z \\
&\quad +2(ra - 2sb)bw_x + 4(a(ra - sb) - rb)bw_z\Big), \\
r_t &= \frac{1}{r^2 - 2rsa + 2s^2b}\Big((vs - 2wr)ra_x - rsa_y \\
&\quad -(2w(ra - sb) - us)ra_z + (2wr - vs)sb_x \\
&\quad +s^2b_y + (wr^2 - us^2)b_z + (sa - r)su_x + (2sb - ra)su_z \\
&\quad +(r - sa)rv_x + (ra - 2sb)rv_z \\
&\quad +(2sb - ra)rw_x - 2(a(ra - sb) - rb)rw_z\Big), \\
s_t &= w_x + aw_z + wa_z, \\
u_t &= ar_x + 2br_z - sb_x, \\
v_t &= r_x + ar_z + as_x + 2bs_z - sa_x + sb_z, \\
w_t &= s_x + as_z + sa_z.
\end{aligned}
\tag{52}
$$

Theorem 14 ([S19]) *The seven-component 4D evolutionary system (52) is integrable since it admits a Lax pair of the form (44) with algebraic Lax functions f and g given by*

$$\begin{aligned} f &= u + vp + wp^2 + (r + sp)\sqrt{p^2 + 2ap + 2b}, \\ g &= \sqrt{p^2 + 2ap + 2b}, \end{aligned} \tag{53}$$

that is,

$$\begin{aligned} \chi_y &= \frac{1}{g}\Bigg(\big(2sp^2 + (r + 3sa + 2wg)p + ra + vg + 2sb\big)\chi_x \\ &+\big(-sp^3 - (wg + sa)p^2 + pra + 2rb + ug\big)\chi_z \\ &+\big(s_z p^4 + (2as_z + sa_z + r_z + gw_z - s_x)p^3 \\ &+((v_z - w_x)g + 2bs_z + ra_z + sb_z \\ &-2as_x - sa_x - r_x + 2ar_z)p^2 \\ &+((u_z - v_x)g + rb_z - ra_x - sb_x - 2bs_x - 2r_x a + 2br_z)p \\ &-rb_x - gu_x - 2br_x\big)\chi_p\Bigg), \\ \chi_t &= \frac{1}{g}\Bigg((p + a)\chi_x + (ap + 2b)\chi_z + (a_z p^2 + p(b_z - a_x) - b_x)\chi_p\Bigg), \end{aligned} \tag{54}$$

and a nonlinear Lax pair of the form (49) with f and g given by (53):

$$\begin{aligned} \psi_y &= u\psi_z + v\psi_x + w\psi_x^2/\psi_z + (r\psi_z + s\psi_x)\sqrt{(\psi_x/\psi_z)^2 + 2a\psi_x/\psi_z + 2b}, \\ \psi_t &= \psi_z\sqrt{(\psi_x/\psi_z)^2 + 2a\psi_x/\psi_z + 2b}. \end{aligned}$$

List of publications constituting the dissertation

[SV] A. Sergyeyev, R. Vitolo, Symmetries and conservation laws for the Karczewska–Rozmej–Rutkowski–Infeld equation, *Nonlin. Analysis: Real World Applications* **32** (2016), 1–9, arXiv:1511.03975.

[S01] A. Sergyeyev, Symmetries and integrability: Bakirov system revisited, *J. Phys. A: Math. Gen.* **34** (2001), no. 23, 4983–4990.

[PS] R.O. Popovych, A. Sergyeyev, Conservation laws and normal forms of evolution equations, *Phys. Lett. A* **374** (2010), 2210–2217, arXiv:1003.1648.

[S04] A. Sergyeyev, A simple way of making a Hamiltonian system into a bi-Hamiltonian one, *Acta Appl. Math.* **83** (2004), no. 1-2, 183–197, arXiv:nlin/0310012.

[S05] A. Sergyeyev, A strange recursion operator demystified, *J. Phys. A: Math. Gen.* **38** (2005), no.15, L257–L262, arXiv:nlin/0406032.

[SD] A. Sergyeyev, D. Demskoi, Sasa–Satsuma (complex modified Korteweg–de Vries II) and the complex sine-Gordon II equation revisited: recursion operators, nonlocal symmetries and more, *J. Math. Phys.* **48** (2007), paper 042702, 11 pp, arXiv:nlin/0512042.

[MS] O.I. Morozov, A. Sergyeyev, The four-dimensional Martínez Alonso–Shabat equation: reductions and nonlocal symmetries, *J. Geom. Phys.* **85** (2014), 40–45, arXiv:1401.7942.

[S17] A. Sergyeyev, A Simple Construction of Recursion Operators for Multidimensional Dispersionless Integrable Systems, *J. Math. Anal. Appl.* **454** (2017), no. 2, 468–480, arXiv:1501.01955.

[S18] A. Sergyeyev, New integrable (3+1)-dimensional systems and contact geometry, *Lett. Math. Phys.* **108** (2018), no. 2, 359–376, arXiv:1401.2122.

[S19] A. Sergyeyev, Integrable (3+1)-dimensional system with an algebraic Lax pair, *Appl. Math. Lett.* **92** (2019), 196–200, arXiv:1812.02263.

Other references

[1] M.J. Ablowitz, P.A. Clarkson, Solitons, nonlinear evolution equations and inverse scattering, Cambridge Univ. Press, Cambridge, 1991.

[2] L.M. Alonso, A.B. Shabat, Hydrodynamic reductions and solutions of a universal hierarchy, Theor. Math. Phys. 104 (2004), 1073–1085, arXiv:nlin/0312043.

[3] V.I. Arnold, Mathematical methods of classical mechanics. Springer-Verlag, New York, 1989.

[4] M. Atiyah, On the work of Simon Donaldson, in: Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986), 3–6, Amer. Math. Soc., Providence, RI, 1987.

[5] M.F. Atiyah, V.G. Drinfeld, N.J. Hitchin, Yu.I. Manin, Construction of instantons. Phys. Lett. A 65 (1978), no.3, 185–187.

[6] I.M. Bakirov, On the symmetries of some system of evolution equations, Preprint, 1991

[7] H. Baran, I.S. Krasil'shchik, O.I. Morozov, P. Vojčák, Nonlocal Symmetries of Integrable Linearly Degenerate Equations: A Comparative Study, Theor. Math. Phys. 196 (2018), 1089–1110.

[8] F. Beukers, J.A. Sanders, J.P. Wang, One symmetry does not imply integrability. J. Differential Equations 146 (1998), no. 1, 251–260.

[9] M. Błaszak, Multi-Hamiltonian theory of dynamical systems. Springer-Verlag, Berlin, 1998.

[10] M. Błaszak, A. Sergyeyev, Contact Lax pairs and associated (3+1)-dimensional integrable dispersionless systems, to appear in Nonlinear Systems and Their Remarkable Mathematical Structures, Vol. II, ed. by N. Euler and M. C. Nucci, CRC Press, 2019, arXiv:1901.05181.

[11] D.M.J. Calderbank, B. Kruglikov, Integrability via geometry: dispersionless differential equations in three and four dimensions, preprint arXiv:1612.02753

[12] I. Dorfman, Dirac structures and integrability of nonlinear evolution equations. John Wiley & Sons, Chichester, 1993.

[13] B. Doubrov and E.V. Ferapontov, On the integrability of symplectic Monge–Ampère equations, J. Geom. Phys. 60 (2010), 1604–1616, arXiv:0910.3407

[14] B.A. Dubrovin, V.B. Matveev, S.P. Novikov, Non-linear equations of Korteweg–de Vries type, finite-zone linear operators, and Abelian varieties. Russian Math. Surveys 31 (1976), 59–146.

[15] B.A. Dubrovin, S.P. Novikov, Hamiltonian formalism of one-dimensional systems of the hydrodynamic type and the Bogolyubov-Whitham averaging method. Soviet Math. Dokl. 27 (1983), no. 3, 665–669.

[16] M. Dunajski, Solitons, Instantons and Twistors, Oxford University Press, Oxford, 2010.

[17] E.V. Ferapontov, Nonlocal Hamiltonian operators of hydrodynamic type: differential geometry and applications, in Topics in topology and mathematical physics, Amer. Math. Soc., Providence, RI, 1995, 33–58.

[18] E.V. Ferapontov, K.R. Khusnutdinova, On integrability of (2+1)-dimensional quasilinear systems. Comm. Math. Phys. 248 (2004), 187–206, arXiv:nlin/0305044.

[19] A.S. Fokas, Symmetries and integrability, Stud. Appl. Math. 77 (1987), 253–299

[20] C.S. Gardner, J.M. Greene, M.D. Kruskal, R.M. Miura, Method for Solving the Korteweg-de Vries Equation. Physical Review Letters, 19 (1967), 1095–1097.

[21] G.A. Guthrie, Recursion operators and non-local symmetries, Proc. R. Soc. A 446 (1994), no. 1926, 107–114.

[22] A. Karasu, At. Karasu, S. Yu. Sakovich, A strange recursion operator for a new integrable system of coupled Korteweg-de Vries equations. Acta Appl. Math. 83 (2004), no. 1-2, 85–94.

[23] A. Karczewska, P. Rozmej, E. Infeld, Shallow-water soliton dynamics beyond the Korteweg–de Vries equation, Phys. Rev. E 90 (2014) 012907. arXiv:1401.4859.

[24] A. Karczewska, P. Rozmej, Ł. Rutkowski, A new nonlinear equation in the shallow water wave problem, Phys. Scr. 89 (2014) 054026. arXiv:1401.4261.

[25] I. Kolář, P. Michor, J. Slovák, Natural operations in differential geometry. Springer-Verlag, Berlin, 1993.

[26] I. Krasil'shchik, A. Sergyeyev, Integrability of Anti-Self-Dual Vacuum Einstein Equations with Nonzero Cosmological Constant: An Infinite Hierarchy of Nonlocal Conservation Laws, Ann. Henri Poincaré 20 (2019), no. 8, 2699–2715.

[27] J. Krasil'shchik, A. Verbovetsky, Geometry of jet spaces and integrable systems. J. Geom. Phys. 61 (2011), no. 9, 1633–1674, arXiv:1002.0077.

[28] J. Krasil'shchik, A. Verbovetsky, R. Vitolo, The symbolic computation of integrability structures for partial differential equations. Springer, Cham, 2017.

[29] B. Kruglikov. O. Morozov, Integrable Dispersionless PDEs in 4D, Their Symmetry Pseudogroups and Deformations, Lett. Math. Phys. 105 (2015), 1703–1723.

[30] P.-L. Lions, J.-C. Rochet, Hopf formula and multitime Hamilton-Jacobi equations, Proc. Amer. Math. Soc. 96 (1986), no. 1, 79–84.

[31] J. Liouville, Note sur l'intégration des équations différentielles de la Dynamique, présentée au Bureau des Longitudes le 29 juin 1853, J. Math. Pur. Appl. 20 (1855), 137–138.

[32] F. Magri, A simple model of the integrable Hamiltonian equation. J. Math. Phys. 19 (1978), no. 5, 1156–1162.

[33] S.V. Manakov, P.M. Santini, Integrable dispersionless PDEs arising as commutation condition of pairs of vector fields, J. Phys. Conf. Ser. 482 (2014), 012029

[34] M. Marvan, Another look on recursion operators, in *Differential geometry and applications (Brno, 1995)*, 393–402, Masaryk Univ., Brno, 1996.

[35] A.V. Mikhailov, V.V. Sokolov, Symmetries of differential equations and the problem of integrability, in: Integrability, 19–88, Lecture Notes in Phys., 767, Springer, Berlin, 2009.

[36] O.I. Morozov, The four-dimensional Martínez Alonso–Shabat equation: differential coverings and recursion operators. J. Geom. Phys. 85 (2014), 75–80, arXiv:1309.4993.

[37] O.I. Morozov, A Recursion Operator for the Universal Hierarchy Equation via Cartan's Method of Equivalence, Cent. Eur. J. Math. 12 (2014), 271–283, arXiv:1205.5748.

[38] P.J. Olver, Evolution equations possessing infinitely many symmetries, J. Math. Phys. 18 (1977), no. 6, 1212–1215.

[39] P.J. Olver, Applications of Lie groups to differential equations, 2nd ed., Springer, N.Y., 1993.

[40] N. Sasa, J. Satsuma, New type of soliton solutions for a higher-order nonlinear Schrödinger equation, J. Phys. Soc. Jpn. 60 (1991), 409–417.

[41] W.K. Schief, Self-dual Einstein spaces via a permutability theorem for the Tzitzeica equation, Phys. Lett. A 223 (1996), no. 1-2, 55–62.

[42] T. Shiota, Characterization of Jacobian varieties in terms of soliton equations. Invent. Math. 83 (1986), no. 2, 333–382.

[43] I.A.B. Strachan, Degenerate bi-Hamiltonian structures of hydrodynamic type. Theoret. and Math. Phys. 122 (2000), no. 2, 247–255.

[44] C.H. Taubes, The work of Maxim Kontsevich, in: Proceedings of the International Congress of Mathematicians, Vol. I (Berlin, 1998). Doc. Math. 1998, Extra Vol. I, 119–126.

[45] J.P. Wang, Lenard scheme for two-dimensional periodic Volterra chain. J. Math. Phys. 50 (2009), no. 2, 023506, 25 pp.

[46] V.E. Zakharov, Dispersionless limit of integrable systems in (2+1) dimensions, in Singular Limits of Dispersive Waves, ed. by N.M. Ercolani et al., Plenum Press, N.Y., 1994, 165–174.